COMPRESSIBLE FLOW AERODYNAMICS

(R22A2112)

COURSE FILE

III B. Tech II Semester

(2024-2025)

Prepared By

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Department of Aeronautical Engineering

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

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Maisammaguda, Dhulapally (Post Via. Kompally), Secunderabad – 500100, Telangana State, India.

MRCET VISION

- To become a model institution in the fields of Engineering, Technology and Management.
- To have a perfect synchronization of the ideologies of MRCET with challenging demands of International Pioneering Organizations.

MRCET MISSION

To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become pioneers of Indian vision of modern society.

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- To pursue continual improvement of teaching learning process of Undergraduate and Post Graduate programs in Engineering & Management vigorously.
- To provide state of art infrastructure and expertise to impart the quality education.

PROGRAM OUTCOMES
(PO's)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design / development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multi disciplinary environments.
12. **Life- long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

DEPARTMENT OF AERONAUTICAL ENGINEERING

VISION

Department of Aeronautical Engineering aims to be indispensable source in Aeronautical Engineering which has a zeal to provide the value driven platform for the students to acquire knowledge and empower themselves to shoulder higher responsibility in building a strong nation.

MISSION

The primary mission of the department is to promote engineering education and research. To strive consistently to provide quality education, keeping in pace with time and technology. Department passions to integrate the intellectual, spiritual, ethical and social development of the students for shaping them into dynamic engineers.

QUALITY POLICY STATEMENT

Impart up-to-date knowledge to the students in Aeronautical area to make them quality engineers. Make the students experience the applications on quality equipment and tools. Provide systems, resources and training opportunities to achieve continuous improvement. Maintain global standards in education, training and services.

PROGRAM EDUCATIONAL OBJECTIVES – Aeronautical Engineering

1. **PEO1 (PROFESSIONALISM & CITIZENSHIP):** To create and sustain a community of learning in which students acquire knowledge and learn to apply it professionally with due consideration for ethical, ecological and economic issues.
2. **PEO2 (TECHNICAL ACCOMPLISHMENTS):** To provide knowledge based services to satisfy the needs of society and the industry by providing hands on experience in various technologies in core field.
3. **PEO3 (INVENTION, INNOVATION AND CREATIVITY):** To make the students to design, experiment, analyze, and interpret in the core field with the help of other multi disciplinary concepts wherever applicable.
4. **PEO4 (PROFESSIONAL DEVELOPMENT):** To educate the students to disseminate research findings with good soft skills and become a successful entrepreneur.
5. **PEO5 (HUMAN RESOURCE DEVELOPMENT):** To graduate the students in building national capabilities in technology, education and research

PROGRAM SPECIFIC OUTCOMES – Aeronautical Engineering

1. To mould students to become a professional with all necessary skills, personality and sound knowledge in basic and advance technological areas.
2. To promote understanding of concepts and develop ability in design manufacture and maintenance of aircraft, aerospace vehicles and associated equipment and develop application capability of the concepts sciences to engineering design and processes.
3. Understanding the current scenario in the field of aeronautics and acquire ability to apply knowledge of engineering, science and mathematics to design and conduct experiments in the field of Aeronautical Engineering.
4. To develop leadership skills in our students necessary to shape the social, intellectual, business and technical worlds.

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III Year B.Tech ANE-II Sem

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(R22A2112) COMPRESSIBLE FLOW AERODYNAMICS

Objectives:

- Study the basic governing equations of compressible flows and its parameters.
- Study the effects of Shock and Expansion waves on aerodynamic characteristics.
- Learn about the experimental methods to study about compressible flows.

Tables: Isentropic, Normal Shock, Oblique Shock, Prandtl Meyer function.

UNIT-I ONE DIMENSIONAL COMPRESSIBLE FLOWS

Review of Thermodynamics. Definition of Compressibility, Stagnation conditions, Speed of sound, Mach number, shock waves. One dimensional flow governing equations. Alternative forms of Energy equations, Normal shock relations with numerical.

UNIT-II OBLIQUE SHOCK AND EXPANSION WAVES

Oblique shock waves. Supersonic flow over a wedge Θ - β - M relations strong and weak shock solutions, regular reflection from a solid boundary. Expansion waves, Prandtl – Meyer Expansion. Shock Expansion theory.

UNIT-III

SUBSONIC COMPRESSIBLE FLOW OVER AIRFOIL

Introduction - Velocity potential equation –small perturbation equation - Prandtl-Glauert compressibility corrections - Critical Mach number with numericals - Drag divergence Mach number - Area rule - Supercritical airfoil.

UNIT – IV

LINEARIZED SUPERSONIC FLOWS AND HYPERSONIC FLOWS

Linearized supersonic pressure coefficient, application to airfoils, lift and drag for flat plate, comparision with shock expansion theory.

Qualitative aspects of hypersonic flows, Newtonian theory, modified Newtonian theory, lift and drag.

UNIT- V

FLOW THROUGH NOZZLES AND VARIABLE AREA DUCTS

Quasi one dimensional flow, Area-velocity relation, Isentropic flow through Convergent – Divergent nozzles. Choked flow conditions. Under and Over expansion conditions. Flow through diffusers – wave reflections from a free boundary. Application to supersonic wind tunnel.

Text Books:

1. Anderson, J .D., Fundamental of Aerodynamics, Mc Graw-Hill International third edition Singapore-2001.

Reference Books:

1. Radhakrishnan, E, E., Gas Dynamics, Prentice Hall of India, 1995.
2. Anderson, J .D., Modern Compressible Flow with Historical Perspective, Mc Graw-Hill International third edition Singapore-2004.

Outcomes:

- Understand the compressible flow parameters shock and expansion wave effecting flow behavior.

- Able to design nozzle, diffuser and variable area ducts to obtain required aerodynamic outputs.
- Able to understand hypersonic flows.

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III B.TECH II SEMESTER – AERONAUTICAL ENGINEERING
HIGH SPEED AERODYNAMICS - II (R15)
MODEL PAPER – I
MAXIMUM MARKS: 75

PART A **Max Marks: 25**

i. All questions in this section are compulsory
ii. Answer in TWO to FOUR sentences.

1. Define compressible and incompressible flows. (2 M)
2. Using a neat sketch show the shock pattern in supersonic flow regime and state the changes in the flow after a shock wave. (3 M)
3. Define one – dimensional flow and quasi one – dimensional flows. Give suitable examples for each. (3 M)
4. For a calorically perfect gas prove that the square of mach number is proportional to ratio of kinetic and internal energy. (2 M)
5. Using necessary assumptions prove that the tangential component of flow velocity is preserved across an oblique shock wave. (3M)
6. Using neat sketch, define Mach reflection. (2 M)
7. Define the terms choking, over expanded, under expanded nozzles. (2 M)
8. Give the relation between incompressible pressure/force coefficient and compressible pressure/force coefficients in a linearised subsonic flow. (3 M)
9. Formulate finite difference method. (2 M)
10. Define truncation error and round – off error. (3 M)

PART B **Max Marks: 50**

i. Answer only one question among the two questions in choice.
ii. Each question answer (irrespective of the bits) carries 10M.

11. A pressure vessel has a volume of 10 m^3 is used to store a high pressure air for operating a supersonic wind tunnel. If the air pressure and temperature inside the vessel are 20 atm and 300 K respectively, calculate
    a. Mass of the air stored inside the vessel
    b. Total energy of the gas stored inside the vessel
    c. If the gas in the vessel is heated, the temperature rises to 600 K calculate the change in entropy of the air inside the vessel.

OR

12. a. State second law of thermodynamics and derive the relations for calculating the change in entropy.
    b. Derive the isentropic flow relations.

In either, explain the nomenclature used clearly.

13. Starting from the steady flow one dimensional energy equation derive the various alternative forms of energy equations. Explain all the symbols used clearly.

OR

14. For the flow across a normal shock
    a. Prove that $a^{*2} = u_1u_2$ (Prandtl's relation)
    b. The Mach number behind a normal shock is always subsonic.
    c. The total temperature across a normal shock wave is constant

15. Making necessary assumptions/using required conditions derive the relation between flow deflection angle, shock angle and upstream Mach number (θ-β-M)

OR

16. a. Derive the governing equation for Prandtl – Meyer expansion flow.
    b. Consider the flow past an expansion corner of angle 30°. The upstream Mach number, pressure and temperature are given by 2, 3 atm and 400 K respectively. Calculate the downstream Mach number, pressure, temperature, total temperature and total pressure.

17. Consider a flat plate at with chord length *c* at an angle of attack α to a supersonic free stream mach number M_∞. Let *L* and *D* be lift and drag per unit span S is plan-form area of the plate per unit span, *S* = *c(1)*. Using linearised theory, derive the following expressions for lift and drag coefficients.

$$C_L = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}; \; C_D = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$$

OR

18. Consider a rocket engine burning Hydrogen and oxygen. The combustion chamber pressure and temperature are 25 atm and 3571 K, respectively. The molecular weight of the chemically reacting gas in the combustion chamber is 16. The pressure at the exit of the convergent – divergent rocket nozzle is 1.174 x 10^2 atm. The throat area is 0.4 m^2. Assuming a calorically perfect gas, calculate a) the exit Mach number , b) the exit velocity , c) the mass flow through the nozzle , and d) the area at the exit

19. a. Explain about similarities of flow to be satisfied for Model testing.
    b. Illustrate the flow over a delta wing in supersonic flow.

OR

20. Write a short note on Hotwire anemometer.

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III B.TECH I SEMESTER – AERONAUTICAL ENGINEERING
HIGH SPEED AERODYNAMICS - II (R15)
MODEL PAPER – II
MAXIMUM MARKS: 75

PART A **Max Marks: 25**

i. All questions in this section are compulsory
ii. Answer in TWO to FOUR sentences.

1. State first and second law of Thermodynamics. Define entropy, internal energy and enthalpy. (3M)
2. Calculate the isothermal compressibility of air at a pressure of 0.5 atm. (2 M)
3. Define characteristic speed of sound and stagnation speed of sound. (3 M)
4. Give the relations between characteristic properties and stagnation properties of a flow.(2 M)
5. Define shock strength and classify strong and weak shocks. (2 M)
6. State the advantages of graphical representation of the solution of a flow problem. (3 M)
7. Using neat schematic sketch, explain the application of nozzles. (3 M)
8. Define critical Mach number and drag – divergence Mach number. (2 M)
9. Write about advantages of delta wing. (2 M)
10. Sketch the surface stream lines on a cone at an AoA. (3 M)

PART B **Max Marks: 50**

i. Answer only one question among the two questions in choice.
ii. Each question answer (irrespective of the bits) carries 10M.

11. a. Define speed of sound. Derive the expressions for speed of sound I terms of pressure, density and temperature. **(5 M)**
b. Define thermally perfect and calorically perfect gases. Give the equation of state for calorically and thermally perfect gases. **(5 M)**

OR

12. a. Air flows through a duct. The pressure and temperature at station 1 are 0.7 atm and 30^0C, respectively. At a second station, the pressure is 0.5 atm. Calculate the temperature and density at the second station. Assume the flow to be isentropic.
b. State the limitations of air as a perfect gas.
c. Air at 30^0C is compressed isentropically to occupy a volume which is 1/30 of its initial volume. Assuming air as an ideal gas, determine the final temperature.

13. Using energy equation, derive the relation between static properties and stagnation properties of a flow making necessary assumptions.

OR

14. a. Derive the relation between total pressures across normal shock waves. Explain all the symbols used clearly. (5 M)

    b. A re-entry vehicle is at an altitude of 15,000 m and has a velocity of 1850 m/s. a bow shock wave envelops the vehicle. Neglecting disassociation, determine the static and stagnation pressure just behind the shock wave on the vehicles center line where the shock is assumed to be normal shock. Assume that air behaves as perfect gas with $\gamma = 1.4$ and R = 287 J/kg - K. (5 M)

15. A uniform supersonic stream with $M_1 = 3.0$, $p_1 = 1$ atm and $T_1 = 288$ K encounters a compression corner which deflects the flow stream by an angle of 20^0C. Calculate the shock wave angle and p_2, $T_{2,}$, M_2, p_{o2}, T_{02} behind the shock wave. All the symbols used are standard. Comment on the result if the deflection angle is increased keeping Mach number constant and the Mach number is increased with deflection angle constant, while the remaining parameters are the same.

OR

16. A flat plate is kept at 15^o angle of attack to a supersonic flow at Mach number 2.4. Solve the flow field around the plate and determine the inclination of slipstream direction using shock expansion theory.

17. a) Define linearization. Obtain an expression for linearized pressure coefficient.

    b) Obtain an expression for pressure coefficient for a linearized subsonic flow over a two dimensional profile.(Prandtl-Glauert rule).

    c) The low-speed lift coefficient for an NACA 2412 airfoil at an angle of attack of 4^0 is 0.65. Using the Prandtl-Glauert rule, calculate the lift coefficient for $M_\infty = 0.7$.

OR

18. a) What is diffuser? Sketch a nozzle with conventional supersonic diffuser

    b) A supersonic wind tunnel is designed is designed to produce flow at Mach 2.4. at standard atmospheric conditions. Calculate (i) the exit to throat area ratio of the nozzle (ii) Reservoir pressure and temperature.

19. Describe briefly about components of wind tunnel and flow measurement devices.

OR

20. Write a short note on Laser Doppler anemometer.

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III B.TECH I SEMESTER – AERONAUTICAL ENGINEERING
HIGH SPEED AERODYNAMICS - II (R15)
MODEL PAPER – III
MAXIMUM MARKS: 75

PART A **Max Marks: 25**

i. All questions in this section are compulsory
ii. Answer in TWO to FOUR sentences.

1. Define isentropic flow. State the relation between flow properties in an isentropic flow. (2M)
2. At the nose of the missile in flight, the pressure and temperature are 5.6 atm and 850^0C, respectively. Calculate the density and specific volume. (3 M)
3. What are the governing equations for steady one – dimensional flow? (2 M)
4. For a flow through a variable area duct, give the relation between Area and velocity of the flow. What are the assumptions made in deriving this equation (3 M)
5. Focus on the formation of three – dimensional shock waves. (2 M)
6. State the difference between flow over wedges and cones. (3 M)
7. Give the governing equations for quasi 1- D flow. (2 M)
8. Give the three echelons of transonic inviscid flow theory. (3 M)
9. What is kinematic similarity of flow. (2 M)
10. What types of experiments are carried out by suing wind tunnel? (3 M)

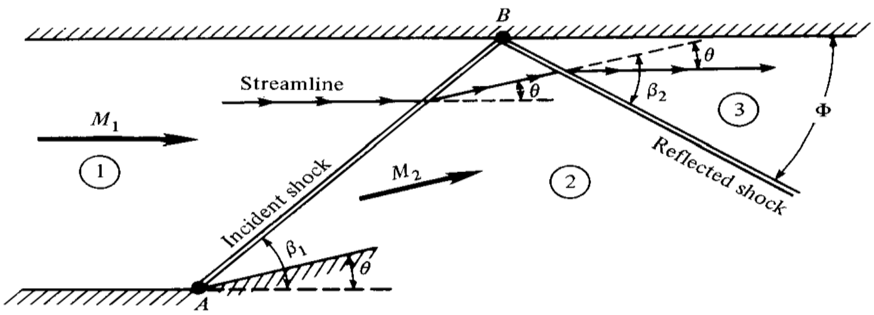
PART B **Max Marks: 50**

i. Answer only one question among the two questions in choice.
ii. Each question answer (irrespective of the bits) carries 10M.

11. Air flows isentropically through a nozzle. If the velocity and the temperature at the exit of the nozzle are 390 m/s and 28^{o}C, respectively, determine the Mach number and Stagnation temperature at the exit. What will be the Mach number just upstream of a station where the temperature is 92.5^{o}C.

OR

12. Derive the normal relations for a perfect gas. Make necessary assumptions and explain the nomenclature.
13. Consider a supersonic flow at Mach 2.8 with a static pressure and temperature of 1 atm and 519^0 R, respectively. The flow passes over a compression corner with a deflection angle of 16^0. The oblique shock generated at the corner propagates into the flow, and is incident on a horizontal wall. Calculate the angle Φ made by the reflected shock wave with respect to the wall, and the Mach number, pressure and temperature behind the reflected shock. Assume that the flow is parallel to the horizontal after moving across the reflected shock.



OR

14. a. Write about shock polar and pressure deflection diagrams.
    b. Explain about prandtl-meyer expansion waves.

15. a) Define Area rule and its importance in designing supersonic aircraft.
    b) Define critical Mach number. Obtain an expression for pressure coefficient at critical Mach number.

OR

16. a) Derive the linearised supersonic flow governing equation.
    b) At $\alpha = 0^0$, the minimum pressure coefficient for an NACA 0009 airfoil in low-speed flow is -0.25. Calculate the critical Mach number for this airfoil using Prandtl-Glauert rule and Karman-Tsien rule.
17. Explain about the method of characteristics for supersonic wind tunnel design.

OR

18. Explain about Quasi one dimensional flow and the area mach relation with over and under expanded flows.
19. Write a short note on Blow down and indraft tunnel layouts and their design features.

OR

20. Write a short note on advantages and disadvantages of wind tunnel.

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III B.TECH I SEMESTER – AERONAUTICAL ENGINEERING
HIGH SPEED AERODYNAMICS - II (R15)
MODEL PAPER – IV
MAXIMUM MARKS: 75

PART A **Max Marks: 25**

i. All questions in this section are compulsory
ii. Answer in TWO to FOUR sentences.

1. Define stagnation conditions and characteristic conditions. (2M)
2. Give the 3 basic governing equations of fluid flow . (3 M)
3. Give the laplace equation in terms of speed of sound. (2 M)
4. Show that the mass flow rate across a stream tube in compressible flow field is inversely proportional to its sectional area. (3 M)
5. Consider a supersonic flow at Mach 2.8 over a compression corner with a deflection angle of 15^0. If the deflection angle is doubled, what is the increase in shock strength? Is it also doubled? Comment. (2 M)
6. Give Prandtl – Meyer function and its significance. (3 M)
7. Define linearization. Give the small perturbation equation. (2 M)
8. Give the expression for C_{pcr} and necessary deductions. (3 M)
9. Define region of influence and domain of independence. (2 M)
10. Define dynamic similarity of flows. (3 M)

PART B **Max Marks: 50**

i. Answer only one question among the two questions in choice.
ii. Each question answer (irrespective of the bits) carries 10M.

11. Define Mach number and its importance. Using neat sketches, explain the flow pattern in various flow regimes.

OR

12. a. At a given point in the high speed flow over the airplane wing, the local Mach number, pressure and Temperature are 0.7, 0.2 atm and 250 K respectively. Calculate the values of p_o, T_o, p^*, T^*, a^* at this point. The symbols used are according to the standard convention. (5M)
    b. Consider a normal shock wave in the flow. The upstream conditions are given by M_1=3, p_1 = 1 atm and ρ_1 = 1.23 kg/m^3. Calculate the downstream values p_2, T_2, M_2, u_2, p_{o2}, T_{o2}**.** The symbols used are according to the standard convention. (5M)
13. Using neat sketches, explain the mathematical/graphical procedures for solving the flow problem
    a. When the shocks of opposite families intersect

b. When the shocks of same family intersect

OR

14. Consider an infinitely thin flat plate at an angle of attack of 20^0 in a Mach 3 free – stream. Calculate the magnitude of flow direction angle φ downstream the trailing edge.
15. Derive the linearized pressure coefficient for supersonic flows.

OR

16. A flat plate is kept at 15^o angle of attack to a supersonic flow at Mach number 2.4. Solve the flow field around the plate and determine the inclination of slipstream direction using shock expansion theory.

17. a. Derive the Area – Mach relation for the variable area ducts like a nozzle.

b. Consider the purely subsonic flow in a convergent – divergent duct. The inlet, throat and the exit area are 1 m^2, 0.7 m^2 and 0.85 m^2 respectively. If the inlet Mach and pressure are 0.3 and 0.8 x 10^5 N/m^2, respectively, then calculate: M and p at the throat and exit.

OR

18. Explain about the role of leading edge extension to improve the performance of aircraft at high angle of attack.

19. Write a short note on Non dimensional parameters and explain about its importance in wind tunnel testing.

OR

20. Discuss briefly about schileren flow visualization technique with neat sketch.

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III B.TECH I SEMESTER – AERONAUTICAL ENGINEERING
HIGH SPEED AERODYNAMICS - II (R15)
MODEL PAPER – V
MAXIMUM MARKS: 75

PART A **Max Marks: 25**

i. All questions in this section are compulsory
ii. Answer in TWO to FOUR sentences.

1. Define the terms continuum flow, free – molecular flow and low density or rarefied flows. (3 M)
2. Define the terms Universal gas constant, Gas constant and Boltzmann constant. (2 M)
3. Explain in simple steps, how supersonic stream is generated in a Convergent – divergent nozzle. (3 M)
4. Give the relation of change in entropy of the flow across a normal shock wave. (2 M)
5. Define flow deflection angle, shock angle and mach angle. (2 M)
6. How does an expansion fan or a shock wave behave when they encounter a free boundary? Illustrate the diamond wave pattern using neat sketch. (3 M)
7. State area – rule and define super critical airfoil. (3 M)
8. Give the expressions used for correcting Prandtl – glauret rule. (2 M)
9. Give the expression for pressure coefficient in linearised supersonic flow. (2 M)
10. Define transonic drag. (3 M)

PART B **Max Marks: 50**

i. Answer only one question among the two questions in choice.
ii. Each question answer (irrespective of the bits) carries 10M.

11. a. Define compressibility. (3M)
b. Explain briefly about changes in flow properties due to one dimensional flow with heat addition and friction. (7M)

OR

12. A ramjet flies at 11 km altitude with a flight Mach number of 0.9. In the inlet diffuse, the air is brought to the stagnation condition so that it is stationary just before the combustion chamber. Combustion takes place at constant pressure and a temperature increase of 1500^{0}C takes place. The combustion products are then ejected through the nozzle.
    a. Calculate the stagnation pressure and temperature.
    b. What will be the nozzle exit velocity? (**refer RathaKrishnan, chapter 4)**
13. a. Using neat sketch, explain the change of properties behind a oblique shock wave. (5 M)
b. Upstream of the oblique shock wave M_1 = 3, p_1 = 0.5 atm and T_1 = 200 K. Calculate the effect of wave angle on the down stream properties M_2, p_2, T_2, u_2, ρ_2 for 15 and 30 degrees. (5 M)

OR

14. a. Write short notes on wave reflection from free boundary.

    b. Air flows at Mach 4.0 and pressure 10^5 N/m^2 is turned abruptly by a wall into the flow with a turning angle of 20^o. If the shock is reflected by another wall determine the flow properties M and ρ downstream of the reflected shock.
15. Derive the velocity potential equation.

OR

16. Write a short note on Critical Mach number, Drag divergence number and supercritical airfoil.
17. Derive the expression for mass flow rate of a calorically perfect gas through a choked nozzle.

    Explain the terms used clearly. $\dot{m} = \frac{p_o A^*}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$

OR

18. a. Write a short note on vortex lift and its effect.

    b. Explain briefly about flow behavior over delta wings at high angle of attack.

19. Write a short note on Shadow graph flow visualization technique with neat sketches

OR

20. Discuss briefly about the wind tunnel balances to measure the forces and moments.

Chapter 1
One dimensional compressible flows

1.1 Introduction

The first and foremost point, that is related with high speed aerodynamics is that here the speed of the fluid or air is considerably large, then, how large it is that quantification we will be doing later on, but we can say that, the speed is comparable to the local speed of sound. And then, in order to maintain a flow at very high speed. Obviously, the pressure difference or the pressure changes that will be associated are quite large. Now once, the pressure changes are large, the gases is likely to change its density.

1.2 Compressible flows

Compressible flow is the science of fluid flow where the density change associated with pressure change is significant. Fluid mechanics is the science of fluid flow in which the temperature changes associated with the flow are insignificant. The simple definition of compressible flow as one in which the density is variable requires more elaboration. Consider a small element of fluid of volume v. The pressure exerted on the sides of the element by the neighboring fluid is p. Assume the pressure is now increased by an infinitesimal amount dp. The volume of the element will be correspondingly compressed by the amount dv. Since the volume is reduced, dv is a negative quantity. The compressibility of the fluid, τ , is defined as

$$\tau = -\frac{1}{v}\frac{dp}{dv} \tag{1.1}$$

Fluids such as water are incompressible (i.e density does not change) under normal conditions. But under conditions of high pressure (e.g. 1000 atm) they are compressible. The change in volume is the characteristic feature of a compressible medium under static conditions. Under dynamic conditions, that is when the medium is moving, the characteristic feature for incompressible and compressible flow situations are: the volume flow rate, $\dot{Q} = AV = constant$ at any cross-section of a streamtube for incompressible flow, and the mass flow rate, $\dot{m} = \rho AV = constant$ at any cross-section of a streamtube for compressible flow. Here, A is the cross-sectional area of the streamtube and V and ρ are the the velocity and density of the fluid at that cross-section.

As long as a gas flows at a sufficiently low speed fromone cross-section of a passage to another the change in volume (or density) can be neglected and, therefore, the flow can be treated as incompressible. Although the fluid is compressible, this property may be neglected when the flow is taking place at low speeds. In other

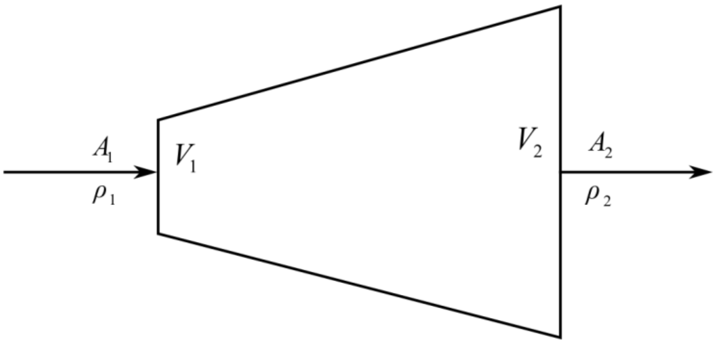


Fig. 1.1: Streamtube

words, although there is some density change associated with every physical flow, it is often possible (for low-speed flows) to neglect it and idealize the flow as incompressible. This approximation is applicable to many practical flow situations, such as low-speed flow around an airplane and flow through a vacuum cleaner.

From the above discussion it is clear that compressibility is the phenomenon by virtue of which the flow changes its density with changes in speed. Now, the question is, what are the precise conditions under which density changes must be considered?

A quantitative measure of compressibility is the volume modulus of elasticity E, defined as

$$E = \frac{\triangle P}{\triangle V/V_i} \tag{1.2}$$

where $\triangle P$ is the change in static pressure, $\triangle V$ is the change in volume, and V_i is the initial volume. For ideal gases, the equation of state is

$$PV = RT \tag{1.3}$$

For isothermal flows, this reduces to

$$PV = P_iV_i = constant \tag{1.4}$$

where P_i is the initial pressure.

The above equation may be written as

$$(P_i + \triangle P)(V_i + \triangle V) = P_iV_i \tag{1.5}$$

Expanding this equation, and neglecting the second-order terms, we get

$$\triangle PV_i + \triangle V P_i = 0 \tag{1.6}$$

Therefore,

$$\triangle P = -P_i \frac{\triangle V}{V_i} \tag{1.7}$$

For gases, from Eqs. 1.2 and 1.7, we get

$$E = P_i \tag{1.8}$$

Hence, by Equ. 1.7, the compressibility may be defined as the volume modulus of the pressure.

Further, By mass conservation, we have $\dot{m} = \rho V = constant$, where $\dot{m}$ is mass flow rate per unit area, V is the flow velocity, and ρ is the corresponding density. This can also be written as

$$(V_i + \triangle V)(\rho_i + \triangle \rho) = \rho_i V_i \tag{1.9}$$

Considering only first-order terms, this simplifies to

$$\frac{\triangle \rho}{\rho_i} = -\frac{\triangle V}{V_i} \tag{1.10}$$

Substituting this into Equ. 1.2, we get

$$\triangle P = E \frac{\triangle \rho}{\rho_i} \tag{1.11}$$

From Equ. 1.11, it can be seen that the compressibility may also be defined as the density modulus of the pressure.

For incompressible flows, by Bernoulli's equation, we have

$$P + \frac{1}{2}\rho V^2 = constant = P_{stag} \tag{1.12}$$

where the subscript "stag" refers to stagnation condition.The above equation may also be written as

$$P_{stag} - P = \frac{1}{2}\rho V^2 \tag{1.13}$$

that is the change of pressure from stagnation to static states is equal to $\frac{1}{2}\rho V^2$. Using equ. 1.11, the above equation can be written as

$$\frac{\triangle P}{E} = \frac{\triangle \rho}{\rho_i} = \frac{\rho_i V_i^2}{2E} = \frac{q_i}{E} \tag{1.14}$$

Here, $q_i = \frac{1}{2}\rho_i V_i^2$ is the dynamic pressure. Equ. 1.14 relates the density change to the flow speed.

The compressibility effects can be neglected if the density changes are very small, i.e. if

$$\frac{\triangle \rho}{\rho_i} \ll 1 \tag{1.15}$$

From equ. 1.14, it is seen that for neglecting compressibility

$$\frac{q}{E} \ll 1 \tag{1.16}$$

For gases, the speed of sound a may be expressed in terms of pressure and density changes as

$$a^2 = \frac{\triangle P}{\triangle \rho} \tag{1.17}$$

Using Equ. 1.11 in the above relation, we get

$$a^2 = \frac{E}{\rho_i} \tag{1.18}$$

Using this Equ. 1.14 changes to

$$\frac{\triangle \rho}{\rho_i} = \frac{\rho_i}{2}\frac{V_i}{E}^2 = \frac{1}{2}\quad\frac{V}{2}^2 \tag{1.19}$$

The ratio *V/a* is called the Mach number *M* .Therefore, the condition of incompressibility for gases becomes

$$\frac{M^2}{2} \ll 1 \tag{1.20}$$

Thus, the criterion determining the effect of compressibility for gases is that the magnitude of theMach numberMshould be negligibly small. Indeed,mathematics would stipulate this limit as $M \to 0$. But Mach number zero corresponds to stagnation state. Therefore, in engineering sciences flows with very small Mach numbers are treated as incompressible. To have a quantification of this limiting value of the Mach number to treat a flow as incompressible, a Mach number corresponding to a 5

$$\frac{\triangle \rho}{\rho} \leq 0.05 \text{ or } 5\% \tag{1.21}$$

that is when $M \leq 0.3$. In other words, the flow may be treated as incompressible when $V \leq 100$ *m/s*, that is when $V \leq 360$ *kmph* under standard sea level conditions.

1.3 Thermodynamics concepts

The kinetic energy per unit mass, $v^2/2$, of a high-speed flow is large. As the flow moves over solid bodies or through ducts such as nozzles and diffusers, the local velocity, hence local kinetic energy, changes. In contrast to low-speed or incompressible flow, these energy changes are substantial enough to strongly interact with other properties of the flow. Because in most cases high-speed flow and compressible flow are synonymous, energy concepts play a major role in the study and understanding of compressible flow. In turn, the science of energy (and entropy) is thermodynamics; consequently, thermodynamics is an essential ingredient in the study of compressible flow.

1.3.1 Thermodynamic Systems

A thermodynamic system is a quantity of matter separated from the ”surroundings” or the ”environment” by an enclosure. The system is studied with the help of measurements carried out and recorded in the surroundings. A thermometer inserted into a system forms part of the surroundings. Work done by moving a piston is measured by, say, the extension of a spring or the movement of a weight in the surroundings. Heat transferred to the system is measured also by changes in the surroundings e.g., heat may be transferred by an electrical heating coil. The electric power is measured in the surroundngs.

Types of systems

Two types of systems can be distinguished. These are referred to, respectively, as closed systems and open systems or control volumes. A closed system or a control mass refers to a fixed quantity of matter, whereas a control volume is a region in space through which mass may flow. A special type of closed system that does not interact with its surroundings is called an Isolated system.

Two types of exchange can occur between the system and its surroundings:

- Energy exchange (heat or work).
- Exchange of matter (movement of molecules across the boundary of the system and surroundings).

Based on the types of exchange, a system can can be called as:

- Isolated Systems: No exchange of matter and energy.
- Closed Systems: No exchange of matter but some exchange of energy.
- Open Systems: Exchange of both matter and energy

If the boundary does not allow heat (energy) exchange to take place it is called adiabatic boundary.

1.3.2 Perfect gas

A perfect gas is one whose individual molecules interact only via direct collisions, with no other intermolecular forces present. For such a perfect gas, the properties p, ρ, and the temperature T are related by the following equation of state

$$p = \rho RT \tag{1.22}$$

where R is the specific gas constant. For air, $R = 287 J/Kg - K$

1.3.3 Internel energy and Enthalpy

Internal energy is the sum of the kinetic and potential energies of the particles that form the system. For an equilibrium system of a real gas where intermolecular forces are important, and also for an equilibrium chemically reacting mixture of perfect gases, the internal energy is a function of both temperature and volume.

Let e denote the specific internal energy (internal energy per unit mass). Then, the enthalpy, h, is defined, per unit mass, as

$$h = e + pv \tag{1.23}$$

In many compressible flow applications, the pressures and temperatures are moderate enough that the gas can be considered to be calorically perfect. Consistent with Equ. 1.22 and the definition of enthalpy is the relation

$$c_p - c_v = R \tag{1.24}$$

where the specific heats at constant pressure and constant volume are defined as

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p \tag{1.25}$$

and

$$c_v = \left(\frac{\partial e}{\partial T}\right)_v \tag{1.26}$$

respectively.

Equ. 1.24 can be deduced into useful forms. Dividing Equ. 1.24 by c_p

$$1 - \frac{c_v}{c_p} = \frac{R}{c_p} \tag{1.27}$$

Defining the heat capacity ratio, $\gamma = c_p/c_v$, Equ. 1.27 becomes

$$1 - \frac{1}{\gamma} = \frac{R}{c_p} \tag{1.28}$$

Solving for c_p,

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{1.29}$$

Similarly, by dividing Equ. 1.27 by c_v, we find that

$$c_v = \frac{R}{\gamma - 1} \tag{1.30}$$

Equations 1.29 and 1.30 hold for a thermally or calorically perfect gas. They will be widely used for treatment of compressible flow.

1.3.4 Laws of Thermodynamics

1.3.4.1 Zeroth law of thermodynamics

This law states that, when system A is in thermal equilibrium with system B and system B is separately in thermal equilibrium with system C then system A and C are also in thermal equilibrium. This law portrays temperature as a property of the system and gives basis of temperature measurement.

1.3.4.2 First law of thermodynamics

Consider a closed system, consisting of a certain amount of gas at rest, across whose boundaries no transfer of mass is possible. Let ∂Q be an incremental amount of heat added to the system across the boundary (by thermal conduction or by direct radiation). Also, let ∂W denote the work done on the system by the surroundings (or by the system on the surroundings). The sign convention is positive when the work is done by the system and negative when the work is done on the system. Owing to the molecular motion of the gas, the system has an internal energy U. The first law of thermodynamics states that the heat added minus work done by the system is equal to the change in the internal energy of the system:

$$\partial Q - \partial W = de \tag{1.31}$$

This is an empirical result confirmed by laboratory experiments and practical experience. In Equ. 1.31, the internal energy U is a state variable (thermodynamic property). Hence, the change in internal energy *de* is an exact differential and its value depends only on the initial and final states of the system. In contrast (the non-thermodynamic properties), ∂Q and ∂W depend on the process by which the system attained its final state from the initial state.

In general, for any given *de*, there are an infinite number of ways (processes) by which heat can be added and work can be done on the system. In the present course of study, we will mainly be concerned with the following three types of processes only.

- Adiabatic process: A process in which no heat is added to or taken away from the system.
- Reversible process: A process which can be reversed without leaving any trace on the surroundings, that is both the system and the surroundings are returned to their initial states at the end of the reverse process.
- Isentropic process: A process which is adiabatic and reversible.

1.3.4.3 The Second Law of Thermodynamics

Let us consider a cold body coming into contact with a hot body. From experience, we can say that the cold body will get heated up and the hot body will cool down. However, Equ. 1.31 does not necessarily imply that this will happen. In fact, the first law allows the cold body to become cooler and the hot body to become hotter as long as energy is conserved during the process. However, in practice this does not happen; instead, the law of nature imposes another condition on the process, a condition that stipulates the direction in which a process should take place. To ascertain the proper direction of a process, let us define a new state variable, the entropy, as follows.

$$ds = \frac{\partial q_{rev}}{T} \tag{1.32}$$

where s is the entropy (amount of disorder) of the system, ∂q_{rev} is an incremental amount of heat added reversibly to the system, and T is the system temperature. The above definition gives the change in entropy in terms of a reversible addition of heat, ∂q_{rev}. Since entropy is a state variable, it can be used in conjunction with any type of process, reversible or irreversible. The quantity ∂q_{rev} is just an artifice; an effective value of ∂q_{rev} can always be assigned to relate the initial and final states of an irreversible process, where the actual amount of heat added is ∂q_{rev}. Indeed, an alternative and probably more lucid relation is

$$ds = \frac{\partial q}{T} + ds_{irrev} \tag{1.33}$$

The above equation applies to all process. It states that the change in entropy during any process is equal to the actual heat added, ∂q, divided by the temperature, $\partial q/T$, plus a contribution from the irreversible dissipative phenomena of viscosity, thermal conductivity, and mass diffusion occurring within the system, ds_{irrev}.These dissipative phenomena always cause an increase in of entropy:

$$ds_{irrev} \geq 0 \tag{1.34}$$

If $ds > 0$, the process is called an irreversible process, and when $ds = 0$, the process is called a reversible process. A reversible and adiabatic process is called an isentropic process. However, in a nonadiabatic process, we can extract heat from the system and thus decrease the entropy of the system.

1.3.5 Entropy Calculation

For a reversible process, Entropy is defined as

$$\partial q = T\ ds \tag{1.35}$$

Using Equ. 1.31, the above equation can be written as

$$T\ ds = de + p\ dv \tag{1.36}$$

$$T\ ds = dh - v\ dp \tag{1.37}$$

The specific heat at constant pressure can be written as

$$c_p = \frac{dh}{dT} \tag{1.38}$$

Substituting the above equation in Equ. 1.37, we get

$$ds = c_p \frac{dT}{T} - \frac{v\ dp}{T} \tag{1.39}$$

Substituting the perfect gas equation, $pv = RT$ into the above equaation, we get

$$ds = c_p \frac{dT}{T} - R \frac{dp}{p} \tag{1.40}$$

Integrating the above equation between states 1 and 2, we get

$$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - R ln \frac{p_2}{p_1} \tag{1.41}$$

Using $de = c_v\ dT$ in the above equation, the change in entropy can be expressed as

$$s_2 - s_1 = c_v ln \frac{T_2}{T_1} + R ln \frac{v_2}{v_1} \tag{1.42}$$

1.3.6 Isentropic relations

An adiabatic and reversible process is called an isentropic process. For an adiabatic process, $\partial q = 0$, and for a reversible process, $ds_{irrev} = 0$. An isentropic process is one for which $ds = 0$, that is the entropy is constant. Important relations for an isentropic process can be obtained from Equ. 1.41 and Equ. 1.42, by setting $s_2 = s_1$. Applying $s_2 = s_1$ in Equ. 1.41 deduces to

$$0 = c_p\ ln \frac{T_2}{T_1} - R\ ln \frac{p_2}{p_1}$$

$$ln \frac{p_2}{p_1} = \frac{c_p}{R}\ ln \frac{T_2}{T_1}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{c_p/R} \tag{1.43}$$

From Equ: 1.29,

$$\frac{c_p}{R} = \frac{\gamma}{\gamma - 1}$$

and substituting Equ. 1.29 into Equ. 1.43, we get

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{1.44}$$

Similarly from Equ. 1.42,

$$0 = c_v\ ln \frac{T_2}{T_1} + R\ ln \frac{v_2}{v_1}$$

$$ln \frac{v_2}{v_1} = -\frac{c_v}{R} ln \frac{T_2}{T_1}$$

$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-c_v/R} \tag{1.45}$$

From Equ. 1.30

$$\frac{c_v}{R} = \frac{1}{\gamma - 1}$$

Substituting the above equation in Equ. 1.45, we get

$$\frac{v_2}{v_1} = \left(\frac{T_2}{T_1}\right)^{-1/(\gamma-1)} \tag{1.46}$$

The above equation can also written as

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{1/(\gamma-1)} \tag{1.47}$$

Summarizing Equ. 1.44 and Equ: 1.47,

$$\boxed{\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}} \tag{1.48}$$

The above Equ. 1.48 relates pressure, density and temperature for an isentropic process. This relation is important and is frequently used in the analysis of compressible flows.

1.4 One-dimensional flow governing equations

Consider the flow through ,I one-dimensional region, a replesented by the shaded area in Fig. ??. This region may be a normal shock wave. or it may be a region with heat addition; in either case. the flow properties change as a function of x as the gas flows through the region. To the left of this region, the flowfield velocity, pressure, temperature, density, and internal energy are u_1, p_l, T_1, p_1, and e_l respectively. To the right of this region, the properties have changed, and are given by u_2, p_2, T_2, p_2, and e_2.

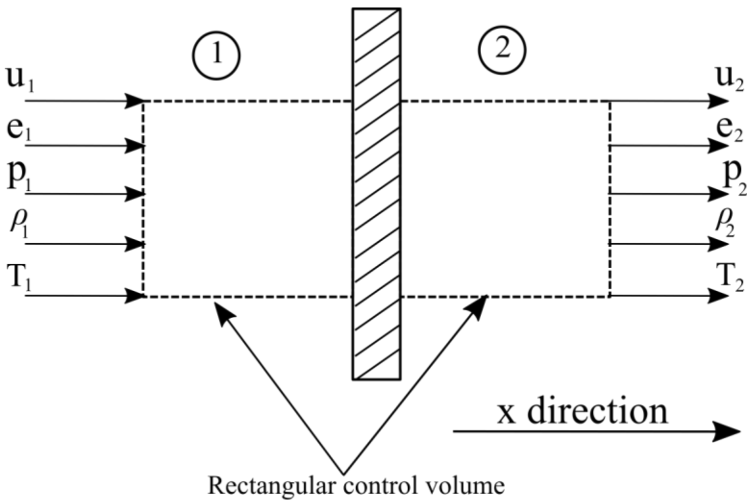


Fig. 1.2: Rectangular control volume for the one-dimensional flow

1.4.0.1 1D continuity equation

The continuity equation is

$$-\oint_S \rho\, V.dS = \frac{\partial}{\partial t}\int_V \rho\, dV \tag{1.49}$$

For the steady flow, the above equation becomes

$$\oint_S \rho\, V.dS = 0 \tag{1.50}$$

Evaluating the surface integral over the left-hand side, where V and dS are parallel but in opposite directions, we obtain $-\rho_1 u_1 A$; over the right-hand side, where V and dS are parallel and in the same direction, we obtain $\rho_2 u_2 A$. The upper and lower horizontal faces of the control volume both contribute nothing to the surface integral because V and dS are perpendicular to each other. The above equations becomes

$$-\rho_1 u_1 A + \rho_2 u_2 A = 0$$

$$\boxed{\rho_1 u_1 = \rho_2 u_2} \tag{1.51}$$

Equ. 1.51 is the continuity equation for steady one-dimensional flow.

1.4.0.2 1D momentum equation

The momentum equation in integral form is

$$\oint_S (\rho V.dS)V + \int_V \frac{\partial(\rho V)}{\partial t} dV = \int_V \rho f dV - \oint_S p\, dS \tag{1.52}$$

For steady flow, the above equation becomes

$$\oint_S (\rho V.dS)V = -\oint_S p\, dS \tag{1.53}$$

The above equation is a vector equation. Since we are dealing in 1D flow, we will consider only the scalar x component of the equation which is

$$\oint_S (\rho V.dS)u = -\oint_S (pdS)_x \tag{1.54}$$

In the above equation, the term $(pdS)_x$ is the x component of the vector $p\, dS$. Solving the above equation over the left and right hand sides of the dashed control volume in Fig. ??, we get

$$\rho_1(-u_1 A)u_1 + \rho_2(u_2 A)u_2 = -(-p_1 A + p_2 A)$$

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2} \tag{1.55}$$

Equ. 1.55 is the momentum equation for steady 1D flow.

1.4.0.3 1D Enery Equation

The energy equation in integral form can be written as

$$\iiint_V \dot{q}\rho dV - \oiint_S pV.dS + \iiint_V \rho(f.V)dV = \iiint_V \frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]dV + \oiint_S \rho\left(e+\frac{V^2}{2}\right)V.dS \quad (1.56)$$

The first term on the left physically represents the total rate of heat added to the gas inside the control volume. For simplicity, let us denote this volume integral by $\dot{Q}$. The third and fourth terms are zero because of zero body forces and steady flow, respectively. Hence, the above equation becomes

$$\dot{Q} - \oiint_S pV.dS = \oiint_S \rho\left(e+\frac{V^2}{2}\right)V.dS \quad (1.57)$$

Evaluating the surface integral over the left and right hand faces of the control volume in Fig. ??, we get

$$\dot{Q} - (-p_1u_1A + p_2u_2A) = -\rho_1\left(e_1+\frac{u_1^2}{2}\right)u_1A + \rho_2\left(e_2+\frac{u_2^2}{2}\right)u_2A$$

Rearranging, we get

$$\frac{\dot{Q}}{A} + p_1u_1 + \rho_1\left(e_1+\frac{u_1^2}{2}\right)u_1 = p_2u_2 + \rho_2\left(e_2+\frac{u_2^2}{2}\right)u_2 \quad (1.58)$$

Dividing the above equation by Equ. 1.51, i.e. dividing the left hand side by ρ_1u_1 and the right hand side by ρ_2u_2,

$$\frac{\dot{Q}}{\rho_1u_1A} + \frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2} \quad (1.59)$$

Considering the 1^{st} term in the above equation, $\dot{Q}$ is the net rate of heat (energy/s) added to the control volume, and ρ_1u_1A is the mass flow (mass/s) through the control volume. Hence, the ratio $\dot{Q}/\rho_1u_1A$ is the heat added per unit mass, q. Also, the definition of enthalpy, $h = e + pv$, Hence, the above equation becomes

$$\boxed{h_1 + \frac{u_1^2}{2} + q = h_2 + \frac{u_2^2}{2}} \quad (1.60)$$

Equ. 1.60 is the energy equation for steady 1D flow.

1.4.1 Speed of Sound

Soundwaves are infinitely small pressure disturbances.The speedwith which sound propagates in a medium is called the speed of sound and is denoted by a. Consider that the sound wave is moving with velocity a through the gas as shown in fig. ??. As the flow pass through the stationary wave front the flow ahead of it moves toward the wave at velocity u with pressure, density, and temperature p, ρ, and T, respectively, and the flow behind it moves away from the wave at velocity $a + da$

with pressure $p + dp$, density $\rho + d\rho$. and temperature $T + dT$.

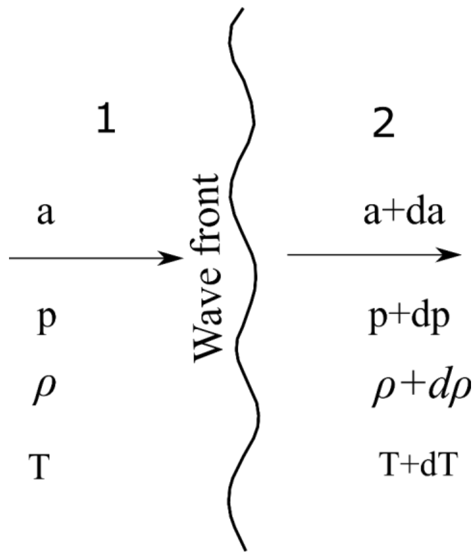


Fig. 1.3: Schematic of a sound wave

The flow through the sound wave is one-dimensional. If regions 1 and 2 are in front of and behind the wave, respectively. Using the continuity equation we can write,

$$\rho a = (\rho + d\rho)(a + da)$$

$$\rho a = \rho a + a d\rho + \rho da + d\rho da \tag{1.61}$$

The product of two infinitesimal quantities $dp\ da$ is very small (2^{nd} order) and hence they can be ignored in the above equation.

$$a = -\rho \frac{da}{d\rho} \tag{1.62}$$

Next the momentum equation yields

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2 \tag{1.63}$$

Ignoring second order (products of differentials) terms as earlier, we get

$$dp = -2a\rho da - a^2 d\rho \tag{1.64}$$

Solving the above equation for da gives,

$$da = \frac{dp + a^2 d\rho}{-2a\rho} \tag{1.65}$$

Substituting Equ. 1.65 into Equ. 1.62, gives

$$a = -rho \ \frac{dp/d\rho + a^2}{-2a\rho} \tag{1.66}$$

Solving the above equation for a^2 gives

$$a^2 = \frac{dp}{d\rho} \tag{1.67}$$

The process inside the sound wave must be isentropic. In turn, the rate of change of pressure with respect to density, dpldp, which appears in Equ. 1.67 is an isentropic change, and Equ. 1.67 can be written as

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \tag{1.68}$$

Equ. 1.68 is the fundamental expression for the speed of sound. It imples that the speed of sound is a direct measure of the compressibility of a gas. Using $\rho = 1/v$, $dp = -dv/v^2$. Hence Equ. 1.68 can be written as

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = -\left(\frac{\partial p}{\partial v}\right)_s v^2 = -\frac{v}{(1/v)(\partial v/\partial p)_s} \tag{1.69}$$

Using the definition of isentropic compressibility, τ_s, we find

$$\boxed{a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\frac{v}{\tau_s}}} \tag{1.70}$$

The above equation confirms the statement ($\tau_s = 0$) implies an infinite speed of sound. For very strong pressure waves, the traveling speed of a disturbance may be greater than that of sound. The pressure can be expressed as

$$p = p(\rho)$$

For an isentropic process of a gas,

$$pv^\gamma = constant$$

Now,

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} \tag{1.71}$$

Hence Equ. 1.70 can be written as

$$\boxed{a = \sqrt{\frac{\gamma p}{\rho}}} \tag{1.72}$$

Using the equation of state, $p/\rho = RT$, the above equation can be written as

$$\boxed{a = \sqrt{\gamma RT}} \tag{1.73}$$

1.4.2 Alternate form of energy equation

Consider Equ. 1.60. Assuming no heat addition, this becomes

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{1.74}$$

Here points 1 and 2 corresponds to regions 1 and 2 identified in Fig. ??. For a calorically perfect gas, $h = c_pT$, the above equations becomes

$$\boxed{c_pT_1 + \frac{u_1^2}{2} = c_pT_2 + \frac{u_2^2}{2}} \tag{1.75}$$

Using Equ. 1.29, the above equation becomes

$$\frac{\gamma RT_1}{\gamma - 1} + \frac{u_1^2}{2} = \frac{\gamma RT_2}{\gamma - 1} + \frac{u_2^2}{2} \tag{1.76}$$

Since $a = \sqrt{\gamma RT}$, the above equation becomes

$$\frac{a_1^2}{\gamma - 1} + \frac{u_1^2}{2} = \frac{a_2^2}{\gamma - 1} + \frac{u_2^2}{2} \tag{1.77}$$

Using $a = \sqrt{\gamma P/\rho}$, the above equation can be written as

$$\boxed{\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{u_1^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{u_2^2}{2}} \tag{1.78}$$

Consider the fluid is brought to Mach 1 at point 2 then flow is sonic in region 2 and suffix 2 is replaced with prefix * in equations representing sonic conditions. On the region 1, $u_1 = u$ and in region 2, $u_2 = a^*$.

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2} \tag{1.79}$$

$$\boxed{\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma + 1}{2(\gamma - 1)}a^{*2}} \tag{1.80}$$

Now consider fluid is brought to rest isentropically i.e $u_2 = 0$, at point 2 representing total conditions denoted by suffix o then, assuming $T_1 = T$ and $u_1 = u$ in region 1 and $u_2 = 0$ and $T_2 = T_o$ Equ. 1.75 changes to

$$c_pT + \frac{u^2}{2} = c_pT_o \tag{1.81}$$

$$\frac{T_o}{T} = 1 + \frac{u^2}{2c^pT} = 1 + \frac{u^2}{2\gamma RT/(\gamma - 1)} = 1 + \frac{u^2}{2a^2/(\gamma - 1)} = 1 + \frac{\gamma - 1}{2}\left(\frac{u}{a}\right)^2$$

Hence,

$$\boxed{\frac{T_o}{T} = a + \frac{\gamma - 1}{2}M^2} \tag{1.82}$$

Equ. 1.82 gives the ratio of total to static temperature in a flow as a function of the Mach number M at that point. Furher, for an isentropic process, Equ. 1.48 holds, such that

$$\frac{p_o}{p} = \left(\frac{\rho_o}{\rho}\right)^{\gamma} = \left(\frac{T_o}{T}\right)^{\gamma/(\gamma-1)} \tag{1.83}$$

Combining Equ. 1.83 and Equ. 1.82, we get

$$\boxed{\frac{p_o}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}} \tag{1.84}$$

$$\boxed{\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}} \tag{1.85}$$

Equ. 1.84 and Equ. 1.85 gives the ratios of total to static pressure and density, respectively, at a point in the flow as a function of Mach number M at that point.

1.5 Flow regimes

The Mach number (M) is defined as the ratio of the speed of an object (or of a flow) to the speed of sound. For instance, in air at room temperature, the speed of sound is about 340 m/s (1,100 ft/s^{-1}. M can range from 0 to ∞, but this broad range falls naturally into several flow regimes. These regimes are subsonic, transonic, supersonic, hypersonic, and hypervelocity flow. The figure below illustrates the Mach number ”spectrum” of these flow regimes.

These flow regimes are not chosen arbitrarily, but rather arise naturally from the strong mathematical background that underlies compressible flow (see the cited reference textbooks). At very slow flow speeds the speed of sound is so much faster that it is mathematically ignored, and the Mach number is irrelevant. Once the speed of the flow approaches the speed of sound, however, the Mach number becomes all-important, and shock waves begin to appear. Thus the transonic regime is described by a different (and much more difficult) mathematical treatment. In the supersonic regime the flow is dominated by wave motion at oblique angles similar to the Mach angle. Above about Mach 5, these wave angles grow so small that a different mathematical approach is required, defining the Hypersonic speed regime. Finally, at speeds comparable to that of planetary atmospheric entry from orbit, in the range of several km/s, the speed of sound is now comparatively so slow that it is once again mathematically ignored in the hyper-velocity regime.

1.6 Normal shock relations

A shock which is perpendicular to flow direction is known as normal shock. It commonly occurs in supersonic flow changes the upstream supersonic flow to subsonic.

Applying one dimensional fluid flow governing equation to flow across normal shock by considering a control volume around it as shown in Fig. 1.4. The basic normal shock equations are directly obtained from equations 1.51, 1.55 and 1.60 with q = 0 (shock wave is adiabatic), we have

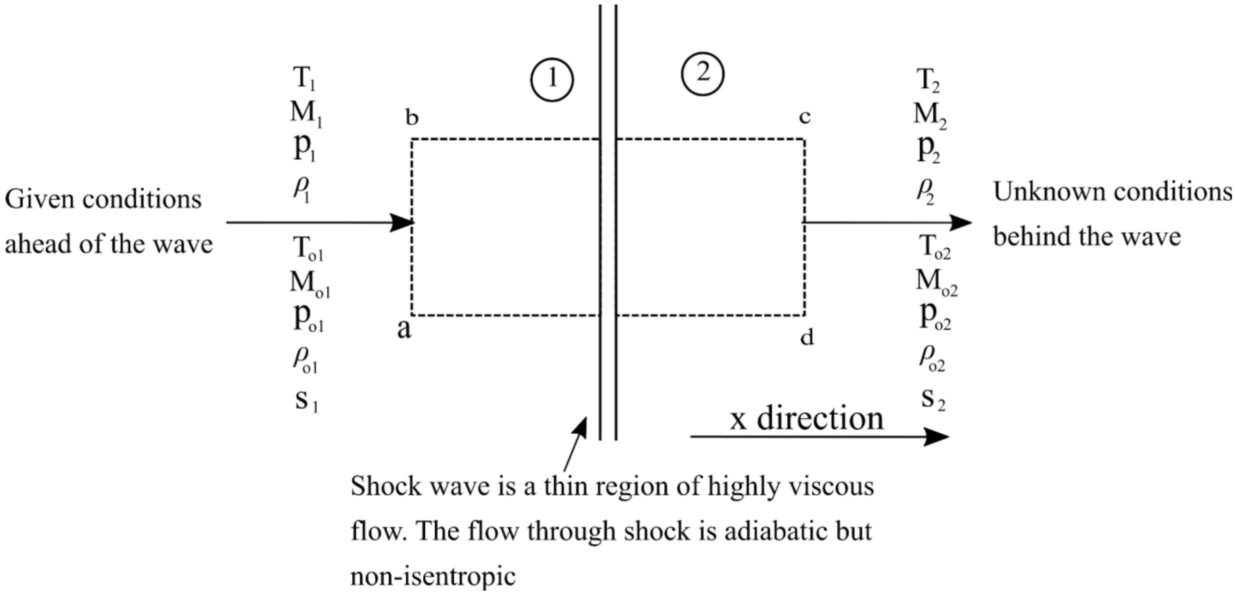


Fig. 1.4: Illustration of flow conditions ahead and behind the normal Shock wave

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

For a calorically perfect gas, we can immediately add the thermodynamic relations

$$p = \rho R T \tag{1.86}$$

$$h = c_p T \tag{1.87}$$

Dividing the momentum equation with continuity equation, we get

$$\frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} = u_2 - u_1 \tag{1.88}$$

Recalling $a = \sqrt{\gamma p/\rho}$, the above equation becomes

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1 \tag{1.89}$$

The alternate form of energy equation, using Equ. 1.60 and Equ. 1.80, yields

$$a_1^2 = \frac{\gamma + 1}{2} a^{*2} - \frac{\gamma - 1}{2} u_1^2 \tag{1.90}$$

and

$$a_2^2 = \frac{\gamma + 1}{2} a^{*2} - \frac{\gamma - 1}{2} u_2^2 \tag{1.91}$$

Since the flow is adiabatic across the shock wave, a^* in the above equations is same constant value. Substituting both of the above alternate form energy equation into Equ. 1.89, we get

$$\frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_1} - \frac{\gamma-1}{2\gamma}u_1 - \frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_2} + \frac{\gamma-1}{2\gamma}u_2 = u_2 - u_1$$

$$\frac{\gamma+1}{2\gamma u_1 u_2}(u_2 - u_1)a^{*2} + \frac{\gamma-1}{2\gamma}(u_2 - u_1) = u_2 - u_1$$

Dividing by $(u_2 - u_1)$,

$$\frac{\gamma+1}{2\gamma u_1 u_2}a^{*2} + \frac{\gamma-1}{2\gamma} = 1$$

Solving for a^*, gives

$$\boxed{a^{*2} = u_1 u_2} \tag{1.92}$$

Equ. 1.92 is called the Prandtl relation, which is useful for intermediate normal shocks. Example, we can obtain

$$1 = \frac{u_1}{a^*} = M_1^* M_2^*$$

$$\boxed{M_2^* = \frac{1}{M_1^*}} \tag{1.93}$$

Here, the flow ahead of a shock wave must be supersonic, i.e. $M_1 > 1$. this implies that $M_1^* > 1$. Thus, from Equ. 1.93, $M_2^* < 1$ and thus $M_2 < 1$, Hence, the *Mach number behind the normal shock is always subsonic*

Now, dividing Equ. 1.80 by u^2, we get

$$\frac{(a/u)^2}{\gamma-1} + \frac{1}{2} = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{a^*}{u}\right)^2$$

$$\frac{(1/M)^2}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}\left(\frac{1}{M^*}\right)^2 - \frac{1}{2}$$

$$\boxed{M^2 = \frac{2}{[(\gamma+1)/M^{*2}]} - (\gamma-1)} \tag{1.94}$$

Equ. 1.94 provides the direct relation between the actual Mach number M and the characteristc Mach number M^*.

Solving for M^* in Equ. 1.94 gives

$$M^{*2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \tag{1.95}$$

Subsituting the Equ. 1.93 into the above equation yields

$$\frac{(\gamma+1)M_2^2}{2+(\gamma-1)M_2^2} = \left[\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}\right]^{-1} \tag{1.96}$$

Solving the above equation for M_2^2

$$\boxed{M_2^2 = \frac{1 + [(\gamma - 1)/2]M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}} \tag{1.97}$$

Equ. 1.97 demonstrates that, for a calorically perfect gas with a constant value of γ, the Mach number behind the shock is a function of only Mach number ahead of the shock. it also shows that when $M_1 = 1$, then $M_2 = 1$. This is the case of an infinitely weak normal shock, which is defined as a Mach wave. In contrast. as M_1 increases above 1, the normal shock becomes stronger and M_2 becomes progressively less than 1.

The other flow properties across a normal shock can be obtained by combining Equ. 1.92 and Equ. 1.51 gives

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_2 u_1} = \frac{u_1^2}{a^{*2}} = M_1^{*2} \tag{1.98}$$

Substituting Equ. 1.95 into the above equation yields

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}} \tag{1.99}$$

To obtain the pressure ratio, the momentum Equ. 1.55 can be written as

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 \tag{1.100}$$

combining the above equation with 1D continuity Equ. 1.51 gives

$$p_2 - p_1 = \rho_1 u_1 (u_1 - u_2) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) \tag{1.101}$$

Dividing the above equation by p_1, and recalling $a_1^2 = \gamma p_1/\rho_1$, we obtain

$$\frac{p_2 - p_1}{p_1} = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right) \tag{1.102}$$

Substitute Equ. 1.99 for u_1/u_2 into the above equation,

$$\frac{p_2 - p_1}{p_1} = \gamma M_1^2 \left(1 - \frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right) \tag{1.103}$$

Simplifying the above equation, we get

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)} \tag{1.104}$$

The temperature ratio from the equation of state $p + \rho RT$ can be written as

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2} \tag{1.105}$$

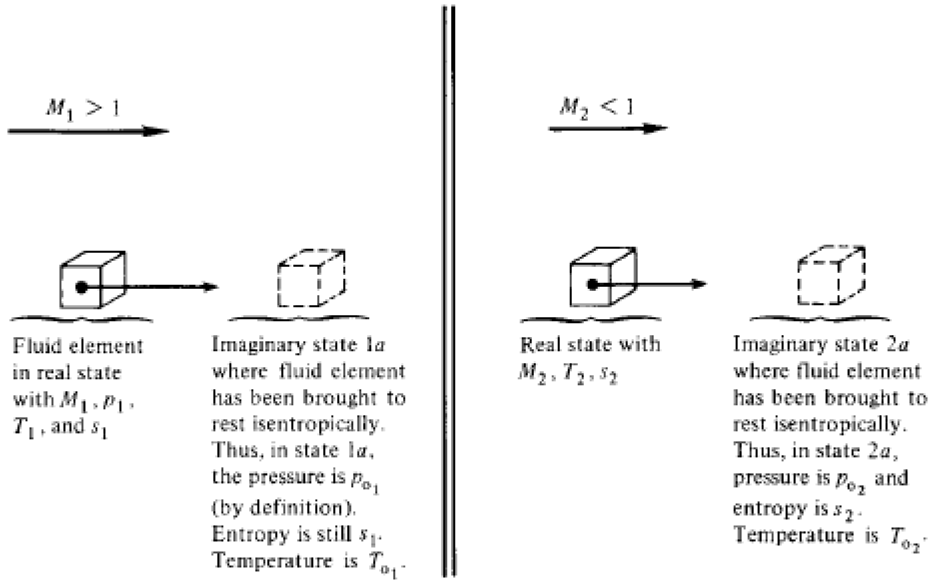


Fig. 1.5: Illustration of total conditions across a normal shock wave

Substituting Equ. 1.104 and Equ. 1.99 into the above equation, gives

$$\boxed{\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\left[\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)M_1^2}\right]} \qquad (1.106)$$

Now, we will study on how the total (stagnation) conditions vary across a normal shock wave.

Fig. 1.5 illustrates the definition of total conditions before and after the shock. In region 1 ahead of the shock, a fluid element is moving with actual conditions of M_1, p_1, T_1 and s_1. Consider in this region the imaginary state la where the fluid element has been brought to rest isentropically. Thus, by definition, the pressure and temperature in state la are the total values p_{ol} ,and T_{o1}, respectively. The entropy at state la is still s_l because the stagnating of the fluid element has been done isentropically. In region 2 behind the shock, a fluid element is moving with actual conditions of M_2, p_2, T_2, and s_2. Consider in this region the imaginary state 2a where the fluid element has been brought to rest isentropically. Here, by definition, the pressure and temperature in state 2a are the total values of p_{ol} and T_{o1} respectively. The entropy at state 2a is still s_2 by definition. The question is now raised how p_{o2} and T_{o2}, behind the shock compare with p_{o2} and T_{o2}, respectively, ahead of the shock. To answer this question, consider equation

$$c_pT_1 + \frac{u_1^2}{2} = c_pT_2 + \frac{u_2^2}{2}$$

From the Equ. 1.81, the total temperature is given by

$$c_pT_o = c_pT + \frac{u^2}{2}$$

Hence,

$$c_pT_{o1} = c_pT_{o2}$$

and thus

$$\boxed{T_{o1} = T_{o2}} \tag{1.107}$$

From Equ. 1.107 it is clear that the total temperature is constant across a stationary normal shock wave.

Now considering Fig. 1.5 and writing Equ. 1.41 between imaginary states 1a and 2a

$$s_{2a} - s_{1a} = c_p \ln\frac{T_{2a}}{T_{1a}} - R\ln\frac{p_{2a}}{p_{1a}} \tag{1.108}$$

However, $s_{2a} = s_2$, $s_{1a} = s_1$, $T_{2a} = T_o = T_{1a}$, $p_{2a} = p_{o2}$, and $p_{1a} = p_{o1}$. Hence the above equation becomes

$$\boxed{s_2 - s_1 = -R\ln\frac{p_{o2}}{p_{o1}}} \tag{1.109}$$

or

$$\frac{p_{o2}}{p_{o1}} = e^{-(s_2-s_1)/R} \tag{1.110}$$

From the above equation, we can see that the ratio of total pressure across the normal shock depends on the M_1 only. Also, because $s_2 > s_1$, the above equations show that $p_{o2} < p_{o1}$. *The total pressure decreases across a shock wave*

1.7 Numerical Problems

1. A pressure vessel that has a volume of $10m^3$ is used to store high-pressure air for operating a supersonic wind tunnel. If the air pressure and temperature inside the vessel are 20 atm and 300 K, respectively. a) what is the mass of air stored in the vessel? b) Total energy of the gas inside the vessel. c) If the gas in the vessel is heated, the temperature rises to 600 K calculate the change in entropy of the air inside the vessel.

   Solution: The pressure, p = 1atm = 101325 Pa

$$\rho = \frac{p}{RT} = \frac{20 \times 101325}{287 \times 300} = 23.46\ kg/m^3$$

   The total mass stored is then

$$m = v\rho = 10 \times 23.46 = 234.6\ kg$$

$$c_v = \frac{R}{\gamma - 1} = \frac{287}{1.4 - 1} = 717.5\ J/kg.K$$

$$e = c_vT = 717.5 \times 300 = 2.153 \times 10^5\ J/kg$$

   The total energy is $E = me = 234.6 \times 2.153 \times 10^5 = 5.05 \times 10^7 J$

From the ideal gas equation $pv = RT$, we can write

$$\frac{p_2}{p_1} = \frac{T_2}{T_1} = \frac{600}{300} = 2$$

The change in entropy is given by

$$s_2 - s_1 = c_p \, ln \frac{T_2}{T_1} - R ln \frac{p_2}{p_1}$$

$$c_p = c_v + R = 717.5 + 287 = 1004.5 \; J/kg.K$$

$$s_2 - s_1 = 1004.5 ln2 - 287 ln2$$

$$s_2 - s_1 = 497.3 \; J/kg.K$$

The total change in entropy is

$$S_2 - S_1 = m(s_2 - s_1) = 234.6 \times 497.3 \; J/kg.K$$

2. Calculate the isothermal compressibility for air at a pressure of 0.5 atm.

   Solution: The compressibility is defined as

$$\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$$

$$v = \frac{RT}{p}$$

   Thus

$$\left(\frac{\partial v}{\partial p}\right)_T = -\frac{RT}{p^2}$$

   Hence

$$\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T = -\frac{p}{RT}\left(-\frac{RT}{p^2}\right) = \frac{1}{p} \tag{1.111}$$

   We can see that, compressibility for a perfect gas is simply the reciprocal of the pressure:

$$\tau_T = \frac{1}{p} = \frac{1}{0.5} = 2 \; atm^{-1} \tag{1.112}$$

3. Air flows through a duct. The pressure and temperature at station 1 are 0.7 atm and 30^0C, respectively. At a second station, the pressure is 0.5 atm. Calculate the temperature and density at the second station. Assume the flow to be isentropic.

   Solution: The ideal gas equation is

$$pv = RT$$

$$\frac{p_2}{p_1} = \frac{T_2}{T_1}$$

$$T_2 = \frac{p_2}{p_1} T_1 = \frac{0.5}{0.7}(30 + 273) = 216.43K$$

$$\rho_2 = \frac{p_2}{RT_2} = \frac{0.5 \times 101325}{287 \times 216.43} = 0.81562$$

4. At the nose of a missile in flight, the pressure and temperature are 5.6 atm and 700 K, respectively. Calculate the density and specific volume.

   Solution: Given,

$$p_o = 5.6\ atm = 5.6 \times 101325 = 567420Pa$$

$$t_o = 700\ K$$

$$\rho_o = \frac{p_o}{RT_o} = \frac{567420}{287 \times 700} = 2.8243\ kg/m^3$$

   The specific volume is

$$v = \frac{1}{\rho_o} = \frac{1}{2.8243} = 0.3541$$

5. At a point in the flow over an F-15 high-performance fighter airplane, the pressure, temperature, and Mach number are 1890 lb/ft2, 450 R, and 1.5, respectively. At this point, calculate T_o, p_o, $T*$, $p*$, and the flow velocity.

   Solution From Table A.1, for M=1.5; p_o/p = 3.671 and T_o/T = 1.45, Thus

$$p_o = 3.671 \times p = 3.671 \times 1890 = 6938\ lb/ft^2$$

$$T_o = 1.45 \times T = 1.45 \times 450 = 652.5\ R$$

   From Table A. 1, for M = 1 .O: p/p_* = 1.893 and T/T_* = 1.2. Keeping in mind that, for our imaginary process where the flow is slowed down isentropically to Mach I , hence defining p*, the total pressure is constant during this process; also, where the flow is slowed down adiabatically to Mach 1 , hence defining T^*, the total temperature is constant. Thus

$$p^* = \frac{p^*}{p_o}\frac{p_o}{p} p = \frac{1}{1.893} \times 3.671 \times 1890 = 3665\ lb/ft^2$$

$$T^* = \frac{T^*}{T_o}\frac{T_o}{T} T = \frac{1}{1.2} \times 1.45 \times 450 = 543.8\ R$$

   Finally, the flow velocity is

$$V = Ma = M\sqrt{\gamma RT} = 1.5 \times \sqrt{1.4 \times 1716 \times 450} = 1560\ ft/s$$

6. A normal shock wave is standing in the test section of a supersonic wind tunnel. Upstream of the wave, $M_1 = 3$, $p_1 = 0.5$ atm, and $T_1 = 200$ K. Find M_2, p_2, T_2. and u_2 downstream of the wave.

   Solution: From A.2, for $M_1 = 3$: $p_2/p_1 = 10.33$, $T_2/T_1 = 2.679$ and $M_2 = 0.4752$. Hence

$$P_2 = \frac{p_2}{p_1} p_1 = 10.33 \times 0.5 = 5.165 \ atm$$

$$T_2 = \frac{T_2}{T_1} T_1 = 2.679 \times 200 = 535.8 \ K$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{1.4 \times 287 \times 535.8} = 464 \ m/s$$

$$u_2 = M_2 a_2 = 0.4752 \times 464 = 220 \ m/s$$

7. A blunt nosed missile is flying at Mach 2 at standard sea level. Calculate the temperature and pressure at the nose of the missile.

   Solution: The nose of the missile is a stagnation point. and the streamline through the stagnation point has also passed through the normal portion of the bow shock wave. Hence, the temperature and pressure at the nose are equal to the total temperature and pressure behind a normal shock. Also, at standard sea level, $T_1 = 288\ K$ and $p_l = 1\ atm = 101325\ pa$.

   From Table A.1, for $M_1 = 2$: $T_{o1}/T_1 = 1.8$ and $p_{o1}/p_1 = 7.824$. Also, for adiabatic flow through a normal shock, $T_{o2} = T_{o1}$, Hence

$$T_{o2} = T_{o1} = \frac{T_{o1}}{T_1} T_1 = 1.8 \times 288 = 518.4 \ K$$

   From Table A.2, for $M_1 = 2$: $p_{o2}/p_{o1} = 0.7209$. Hence

$$p_{o2} = \frac{p_{02}}{p_{o1}} \frac{p_{o1}}{p_1} p_1 = 0.7209 \times 7.824 \times 101325 = 5.72 \times 10^5 \ pa$$

Chapter 2
Oblique Shock and Expansion Waves

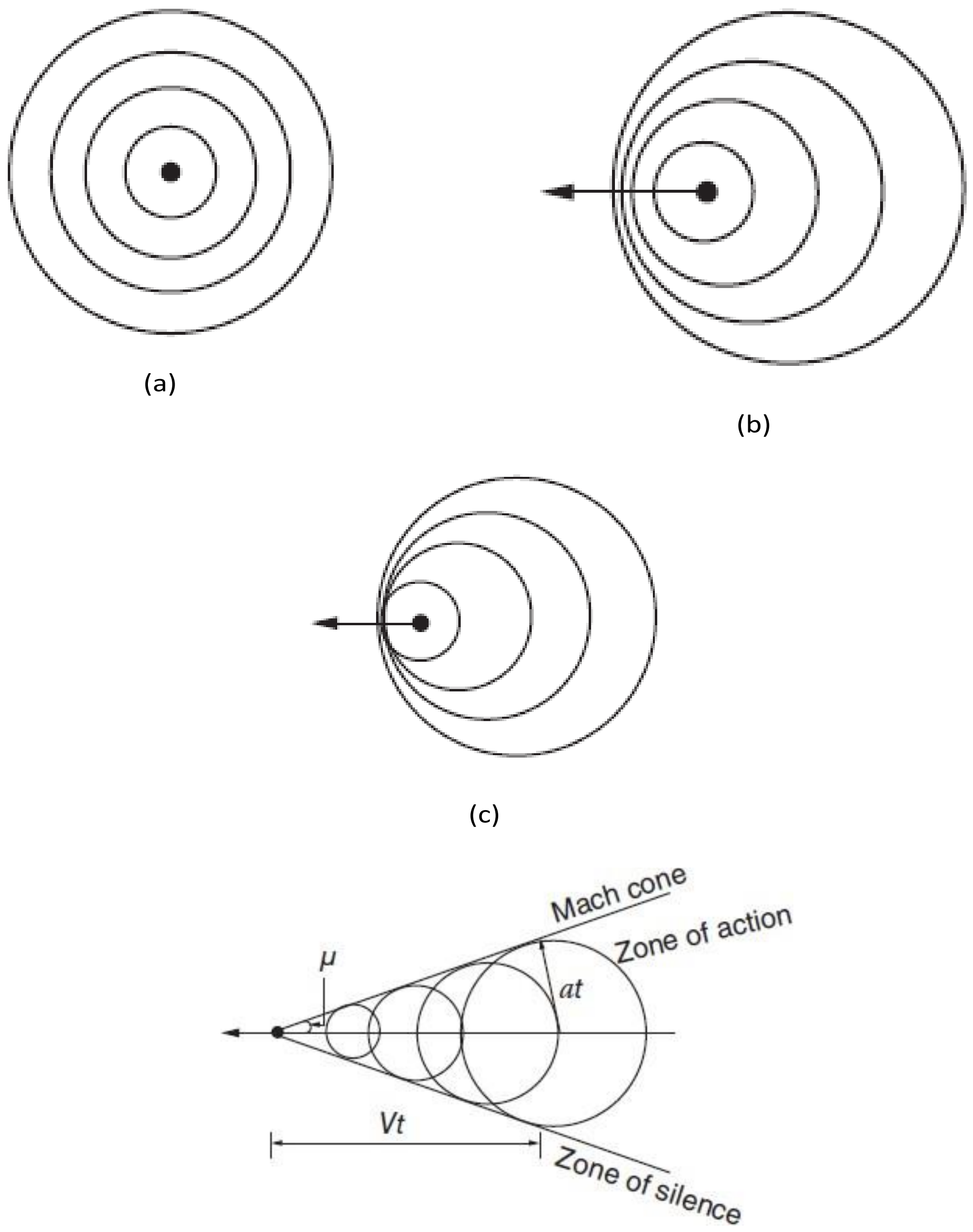
2.1 Introduction

2.1.1 Waves in supersonic flow

The motion of a body in a fluid at rest creates disturbance in the fluid. The disturbances, in general, may not be small. The disturbances in the fluid close to the body are transmitted to other parts of the body and also to the other parts of the fluid through propagation of the waves. The wave motion is compatible with the motion of the body. This wave motion determines the pressures on the body as well as the complete flow field around the body. When the flow is subsonic, it is not essential to consider the wave motion. Particularly, if the motion is steady it is easier to study the motion from a reference system where the body is at rest and the fluid flows over it. However, if the relative wind is supersonic, the waves can not propagate ahead of the immediate vicinity of the body. Thus, the wave system travels with the body and is stationary in the reference system that moves with the body. Limited upstream influence allows the flow to be analyzed or constructed step by step.

Let us examine the propagation of pressure disturbances created by a moving object, shown in Fig. 2.1.In a subsonic flow the disturbance waves reach a stationary observer before the source of disturbance could reach him, as shown in Fig. 2.1(a) and 2.1(b). But in supersonic flows it takes a considerable amount of time for an observer to perceive the pressure disturbance, after the source has passed. This is one of the fundamental differences between subsonic and supersonic flows.Therefore, in a subsonic flow the streamlines sense the presence of any obstacle in the flow field and adjust themselves well ahead of the obstacle and flow around it smoothly. But in a supersonic flow, the streamlines feel the obstacle only when they hit it. The obstacle acts as a source, and the streamlines deviate at the Mach cone, as shown in Fig. 2.1(d) Thus, in a supersonic flow, the disturbance due to an obstacle is sudden and the flow behind the obstacle has to change abruptly.

In Fig. 2.1(d), it is shown that for supersonic motion of an object there is a well-defined conical zone in the flow field with the object located at the nose of the cone, and the disturbance created by the moving object is confined only to the field included inside the cone.The flow field zone outside the cone does not even feel the disturbance. For this reason, von-Karman termed the region inside the cone as the zone of action, and the region outside the cone as the zone of silence.The lines at which the pressure disturbance is concentrated and which generate the cone are called Mach waves or Mach lines.The angle between the Mach line and



(d)

Fig. 2.1: Propagation of disturbance waves *a*) $V = 0$, *b*) $V = a/2$, *c*) $V = a$, *d*) $V > a$.

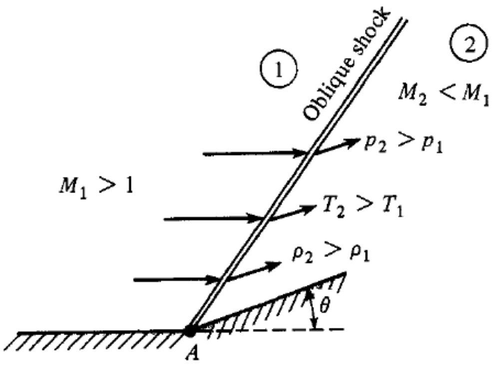
the direction of motion of the body is called the Mach angle μ. From Fig. 2.1(d), we have

$$\sin\mu = \frac{at}{Vt} = \frac{a}{V} \tag{2.1}$$

$$\boxed{\sin\mu = \frac{1}{M}} \tag{2.2}$$

2.2 Oblique shock waves

The normal shock wave, as considered in the previous, is a special case of a more general family of oblique waves that occur in supersonic flow. Oblique shocks usually occur when supersonic flow is turned into itself as shown in Fig. 2.2(a) and Fig. 2.2(b). Stationary shock waves can either be normal or oblique to the flow direction. Necessary relations between the parameters across an oblique shock can be obtained directly from the equations of two-dimensional motion. However, the normal shock results can easily be transformed to obtain the appropriate relations.



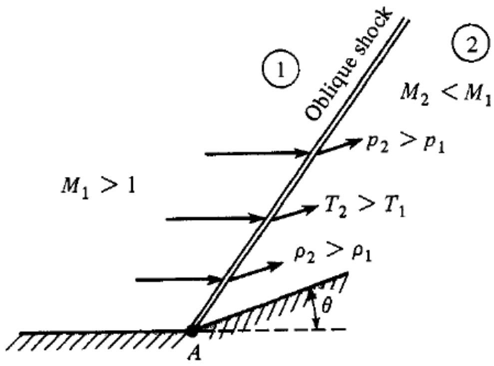


Fig. 2.2: Oblique shock wave produced on *a*) Concave corner *b*) Convex corner

The geometry of flow through an oblique shock is given in Fig. 2.3. The velocity upstream of the shock is V_1 , and is horizontal.The corresponding Mach number is M_1. The oblique shock makes a wave angle β with respect to V_1. Behind the shock, the flow is deflected toward the shock by the flow-deflection angle ϑ. The velocity and Mach number behind the shock are V_2 and M_2, respectively. The components of V_1 perpendicular and parallel, respectively, to the shock are u_l and w_1; the analogous components of V_2 are u_2 and w_2, as shown in Fig. 2.3. Therefore, we can consider the normal and tangential Mach numbers ahead of the shock to be M_{n1}, and M_{t1}, respectively; similarly, we have M_{n2} and M_{t2}, behind the shock.

Consider the control volume drawn between two streamlines through an oblique shock, as illustrated by the dashed lines at the top of Fig. 2.3. Faces *a* and *d* are parallel to the shock wave. Apply the integral continuity equation 1.49. The time derivative in Equ. 1.49 is zero. The surface integral evaluated over faces *a* and *d* of the control volume in Fig 2.3 gives

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

Here, $A_1 = A_2$ = areas of faces *a* and *d*. The faces *b*, *c*, *e*, and *f* of the control volume are parallel to the velocity, and hence contribute nothing to the surface integral (i.e., $V.dS = 0$ for these faces). Thus, the continuity equation for an oblique shock wave is

$$\rho_1 u_1 = \rho_2 u_2 \qquad (2.3)$$

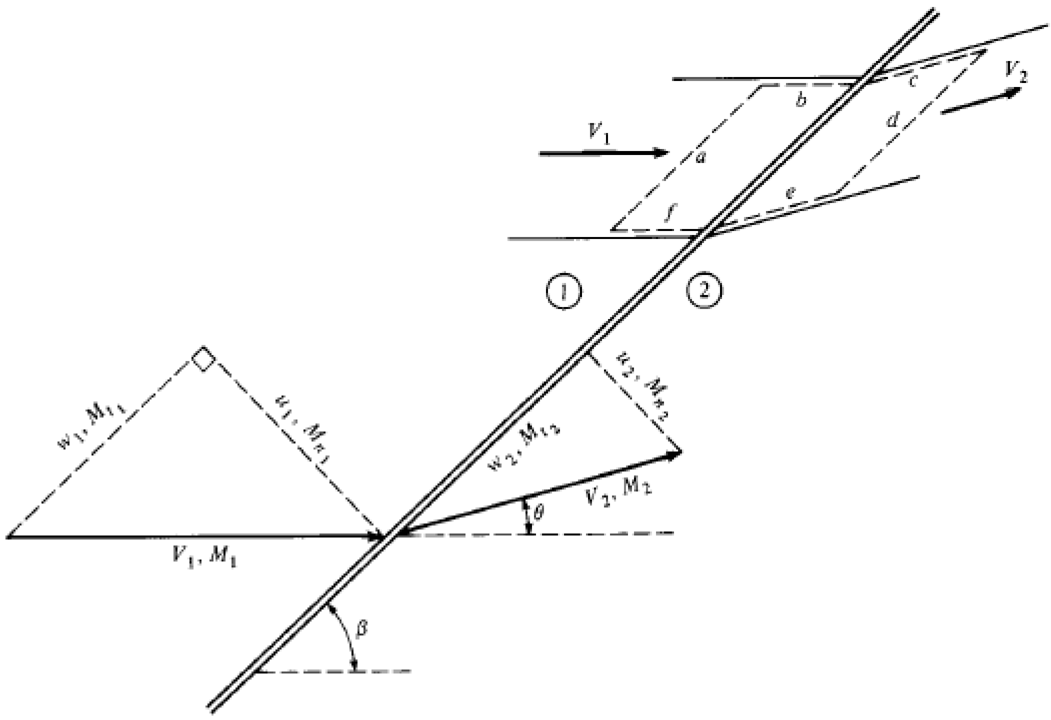


Fig. 2.3: Oblique shock wave geometry

From the integral form of momentum equation Equ. 1.52, considering the equation resolved into two components, parallel and perpendicular to the shock wave in Fig. 2.3 Again, considering steady flow with no body forces, the tangential component of Equ. 1.52 applied to the control surface in Fig. 2.3 yields (noting that the tangential component of $p\ dS$ is zero on faces *a* and *d*, and that the components on *b* cancel those on *f* ; similarly with faces *c* and *e*).

$$(\rho_1 u_1)\, w_1 + (\rho_2 u_2)\, w_2 = 0 \tag{2.4}$$

Deviding Equ. 2.4 by Equ. 2.4, we find that

$$w_1 = w_2 \tag{2.5}$$

The above equation confirms that the *tangential component of flow velocity is preserved across an oblique shock wave*

Now, applying the normal component of Equ. 1.52, we get

$$(-\rho_1 u_1)\, u_1 + (\rho_2 u_2) = -\,(-p_1 + p_2)$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{2.6}$$

Now considering the integral form of energy equation Equ. 1.56. Applied to the control volume in Fig. 2.3 for a steady adiabatic flow with no body forces, it yields

$$-(p_1u_1 + p_2u_2) = -\rho_1\left(e_1 + \frac{V_1^2}{2}\right)u_1 + \rho_2\left(e_2 + \frac{V_2^2}{2}\right)u_2$$

$$\left(h_1 + \frac{V_1^2}{2}\right)\rho_1 u_1 = \left(h_2 + \frac{V_2^2}{2}\right)\rho_2 u_2 \tag{2.7}$$

Dividing the above equation by the continuity Equ. 2.3,

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \tag{2.8}$$

However, in Fig. 2.3 we can see that $V^2 = u^2 + w^2$ and that $w_1 = w_2$. Hence,

$$V_1^2 - V_2^2 = \left(u_1^2 + w_1^2\right) - \left(u_2^2 + w_2^2\right) = u_1^2 - u_2^2 \tag{2.9}$$

Therefore Equ. 2.8 becomes

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{2.10}$$

Observing Equ. 2.3 , Equ. 2.4 and Equ. 2.10, they are similar to the normal shock continuity, momentum and energy equation. Therefore, the changes across an oblique shock wave are governed by the normal component of the free-stream velocity. Furthermore. precisely the same algebra as applied to the normal shock equations in Sec.1.6 , when applied to Equ. 2.3 , Equ. 2.4 and Equ. 2.10. will lead to identical expressions for changes across an oblique shock in terms of the normal component of the upstream Mach number M_{n1}. That is, for an oblique shock wave with

$$M_{n1} = M_1 \sin \beta \tag{2.11}$$

we find the flow properties around an oblique shock wave as

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_{n1}^2}{(\gamma - 1)M_{n1}^2 + 2} \tag{2.12}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_{n1}^2 - 1\right) \tag{2.13}$$

$$M_{n2}^2 = \frac{M_{n1}^2 + [2/(\gamma - 1)]}{[2\gamma/(\gamma - 1)]M_{n1}^2 - 1} \tag{2.14}$$

and

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \tag{2.15}$$

The Mach number behind the oblique shock, M_2, can be found from M_{n2} and the geometry of Fig. 2.3 as

$$M_2 = \frac{M_{n2}}{\sin(\beta - \vartheta)} \tag{2.16}$$

Note: In Sec.1.6, it was found that the changes across a normal shock are a function of only one component - the upstream Mach number.

From the Equ. 2.11 through Equ. 2.15, the changes across an oblique shock are a function of two quantitties - both M_1 and β. We also see that, in reality normal shocks are a just a special case of oblique shocks where $\beta = \pi/2$.

The Equ. 2.16 demonstrates that M_2 cannot be found until the flow deflection angle ϑ is obtained. However, ϑ is a unique function of M_1 and β. From the geometry in Fig. 2.3,

$$\tan\beta = \frac{u_1}{w_1} \tag{2.17}$$

and

$$\tan(\beta - \vartheta) = \frac{u_2}{w_2} \tag{2.18}$$

combining Equ. 2.17 and Equ. 2.18, and noting that $w_1 = w_2$, we get

$$\frac{\tan(\beta - \vartheta)}{\tan\beta} = \frac{u_2}{u_1} \tag{2.19}$$

Combining Equ. 2.19 with Equ. 2.3, Equ. 2.11 and Equ. 2.12, we get

$$\frac{\tan(\beta - \vartheta)}{\tan\beta} = \frac{2 + (\gamma - 1)M_1^2 \sin^2\beta}{(\gamma + 1)M_1^2 \sin^2\beta} \tag{2.20}$$

Solving the above equation by conducting some trigonometric manipulation, the above equation can be expressed as

$$\boxed{\tan\vartheta = 2\cot\beta \, \frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}} \tag{2.21}$$

Equ. 2.21 is called the $\vartheta - \beta - M$ relation, and specifies ϑ as a unique function of M_1 and β. For example $\vartheta = 0$ at $\beta = \frac{\pi}{2}$ and $\beta = \sin^{-1}\frac{1}{M_1}$. Within this range ϑ is positive and must therefore have a maximum. For each value of M_1, there is a maximum value of ϑ. For $\vartheta < \vartheta_{max}$, each value of ϑ and M corresponds to two possible solutions, having different values of β. The larger value of β gives a stronger shock. In the solution with strong shock, the flow becomes subsonic. With weak shock, the flow remains supersonic except for a small range of value of ϑ slightly smaller that ϑ_{max}.

$$\frac{\tan(\beta - \vartheta)}{\tan\beta} = \frac{2 + (\gamma - 1)M_1^2 \sin^2\beta}{(\gamma + 1)M_1^2 \sin^2\beta} \tag{2.22}$$

$$\frac{1}{M_1^2 \sin^2\beta} = \frac{\gamma + 1}{2}\frac{\tan(\beta - \vartheta)}{\tan\beta} - \frac{\gamma - 1}{2}$$

or

$$M_1^2 \sin^2\beta = \frac{\gamma + 1}{2} M_1^2 \frac{\sin\beta \sin\vartheta}{cos(\beta - \vartheta)}$$

$$M_1^2 \sin^2\beta \approx \frac{\gamma + 1}{2} M_1^2 \tan\beta \;\; \vartheta \qquad Forsmallvaluesof\vartheta \tag{2.23}$$

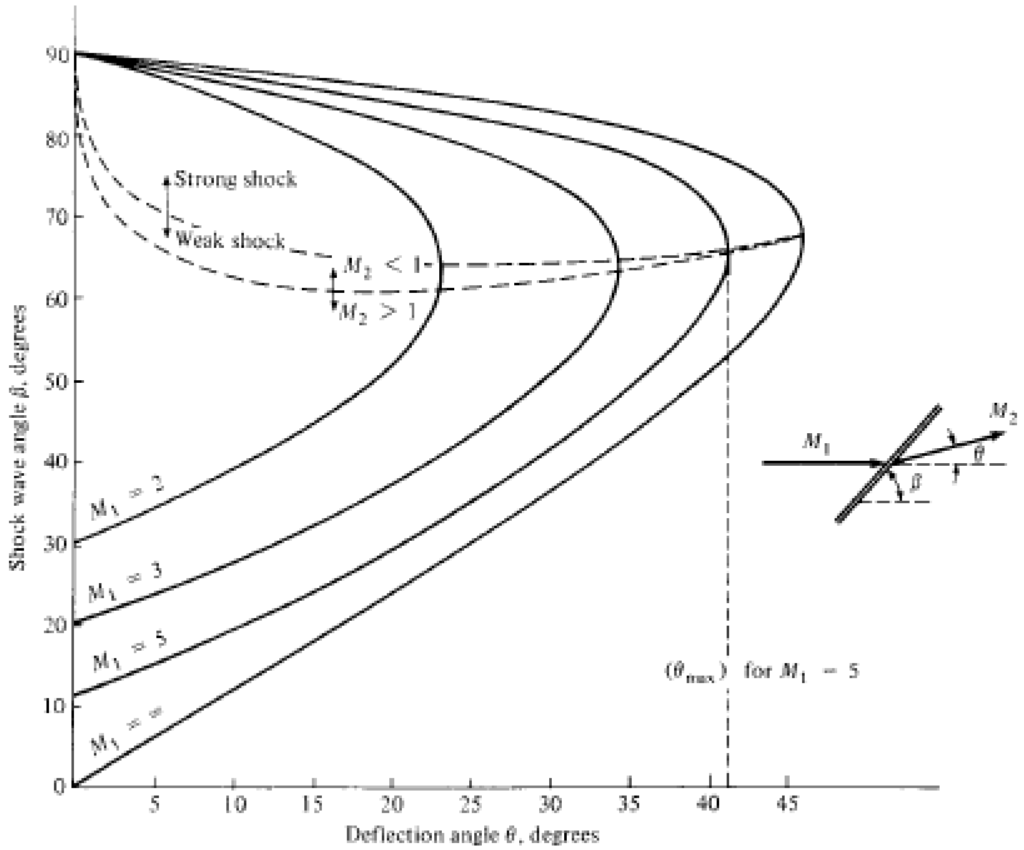


Fig. 2.4: $\vartheta - \beta - M$ curves. Oblique shock properties

2.2.1 Supersonic flow over a wedge

Any streamline in inviscid flow can be replaced by a solid boundary. Thus the oblique shock flow provides the solution to supersonic flow in a corner. For a given values of M_1 and ϑ, the values of β and M_2 are determined. Using symmetry, the flow over a wedge of nose angle 2ϑ is also obtained. The flow on each side of the wedge is determined only by the inclination of the surface on that side. Thus, the wedge need not be symmetric. When the shock waves are attached to the nose, the upper and lower surfaces are independent since there is no influence on the flow upstream of the waves.

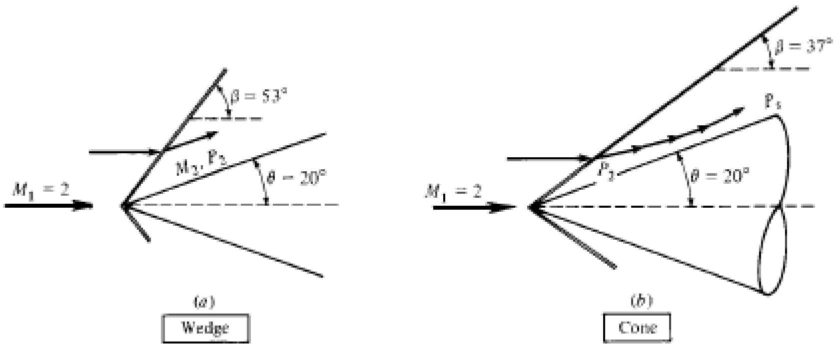


Fig. 2.5: Oblique shock over wedge and cone

2.2.2 Mach Lines

Assuming that downstream flow remain supersonic ($M_2 > 1$), the wave angle β decreases with decrease in wedge angle. When ϑ decreases to zero, β decreases to the limiting value μ, given by

$$M_1^2 \sin^2 \mu - 1 = 0 \quad (or) \quad \mu = \sin^{-1} \frac{1}{M} \tag{2.24}$$

The jump in the flow quantities is then zero and, hence the strength of the wave is zero. The flow is continuous without any disturbance. There is nothing unique about the point where this wave originates; it might be any point in the flow. The angle μ is simply a characteristic angle associated with M_1. It is called the 'Mach angle'. The lines of inclination μ which may be drawn at any point in the flow-field are called 'Mach lines' or 'Mach waves'.

In nonuniform flow μ varies with M and the Mach lines are curved. At any point P in a 2-D flow field, there are always two lines which intersect the streamline at the angle μ. In 3-D flow, the Mach lines or characteristics define a conical surface with vertex at P. A 2-D supersonic flow is always associated with two families of Mach lines called right running and left running characteristics and are often denoted by the labels (+) and (–). Those in the (+) set run to the right of the streamlines and those in the (–) set run to the left. They are called 'characteristics' from the mathematical theory of hyperbolic PDEs. These are analogous to the two families of characteristics that trace the propagation of 1-D waves in the $x - t$ plane. Like the characteristics in the x-t plane, Mach lines have

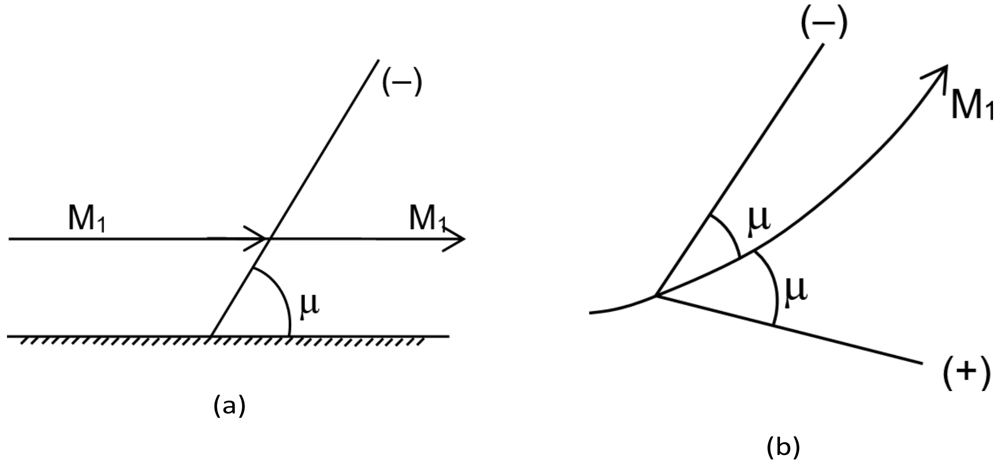


Fig. 2.6: Mach lines *a*) degeneration of Mach line as ϑ approaches 0 *b*) Left and Right running Mach lines at an arbitary point in the flow

a distinguished direction, the direction of flow or the direction of increasing time. This is related to the fact that there is no upstream influence in supersonic flow.

2.2.3 First-order approximation for weak oblique shocks

For small deflection angles ϑ, the oblique shock equations reduce to very simple expressions. The approximate relation that can be used to derive others is

$$M_1^2 \sin^2 \beta(\beta - 1) \approx \frac{\gamma + 1}{2} M_1^2 \tan\beta \quad \vartheta \tag{2.25}$$

For small ϑ, the value of β is close to either $\frac{\pi}{2}$ or μ, depending on whether $M_2 < 1$ or $M_2 > 1$. For $M_2 > 1$, the approximation reduces to

$$M_1^2 \sin^2 \beta - 1 \approx \frac{\gamma + 1}{2} \frac{M_1^2}{\sqrt{M_1^2 - 1}} \vartheta, \qquad as \qquad \tan\beta \approx \tan\mu = \frac{1}{\sqrt{M_1^2 - 1}} \tag{2.26}$$

The pressure is then approximated to

$$\frac{p_2 - p_1}{p_1} = \frac{\triangle p}{p} \approx \frac{\gamma M_1^2}{\sqrt{M_1^2 - 1}} \vartheta \tag{2.27}$$

The changes in other flow quantities are also proportional to the deflection angle ϑ. The change of entropy is proportional to the third power of the shock strength and hence to third power of deflection angle

$$\triangle S \infty \vartheta^3 \tag{2.28}$$

The difference between the wave angle β and the Mach angle μ, to first order accuracy, can be found as follows, Let $\beta = \mu + \varrho$, $\varrho << \mu$ Hence,

$$\sin\beta = \sin(\mu + \varrho) \approx \sin\mu + \varrho\cos\mu \tag{2.29}$$

By definition,

$$\sin\mu = \frac{1}{M_1}, \quad \cot\mu = \sqrt{M_1^2 - 1}$$

Hence,

$$M_1 \sin\beta \approx 1 + \varrho\sqrt{M_1^2 - 1}$$

$$M_1^2 \sin^2\beta \approx 1 + 2\varrho\sqrt{M_1^2 - 1} \tag{2.30}$$

$$M_1^2 \sin^2\beta - 1 \approx 2\varrho\sqrt{M_1^2 - 1} \approx \frac{\gamma+1}{2}\frac{M_1^2}{\sqrt{M_1^2 - 1}}\vartheta \tag{2.31}$$

or

$$\varrho = \frac{\gamma+1}{4}\frac{M_1^2}{M_1^2 - 1}\vartheta \tag{2.32}$$

Hence for a finite deflection angle ϑ, the direction of the wave differs from the Mach direction by an amount ϱ, which is of the same order as ϑ.

The change in flow speed can be obtained as

$$\frac{w_2^2}{w_1^2} = \frac{u_2^2 + v^2}{u_1^2 + v^2} = \frac{\left(\frac{u_2}{v}\right)^2 + 1}{\left(\frac{u_1}{v}\right)^2 + 1} = \frac{\tan^2(\beta - \vartheta) + 1}{\tan^2\beta + 1} = \frac{\cos^2\beta}{\cos^2(\beta - \vartheta)} \tag{2.33}$$

Now,

$$\cos^2\beta = 1 - \sin^2\beta = \tag{2.34}$$

similarly, $\cos^2\beta$ can be obtained by replacing ϱ with $\varrho - \vartheta$. The final result after dropping all terms of order ϑ^2 and higher

$$\frac{w_2}{w_1} \approx 1 - \frac{\vartheta}{\sqrt{M_1^2 - 1}} \quad or \quad \frac{\Delta w}{w_1} = -\frac{\vartheta}{\sqrt{M_1^2 - 1}} \tag{2.35}$$

2.3 Supersonic compression by turning

A shock wave passing through a fluid increases the pressure and density of the fluid. So, shock waves can be used to compress a flow. A simple method for compressing a supersonic flow is to turn it through an oblique shock by deflecting the wall through an angle ϑ. The turn may be subdivided into several segments which make smaller corners of angle $\Delta\vartheta$ so that compression occurs through successive weaker oblique shocks. These shocks divide the field near the wall into segments of uniform flow. In the near wall region each segment of the flow is independent of the next one and may be constructed step by step proceeding downstream. This property of limited upstream influence exists as long as the deflection does not become so great that the flow becomes subsonic. Away from the wall the shocks tend to intersect each other since they are convergent

For each wave in the multiple shock $\Delta p \propto \Delta\vartheta$ and $\Delta s \propto (\Delta\vartheta)^3$. The overall pressure and entropy changes are

$$p_k - p_1 \propto n\Delta\vartheta \;\; \vartheta \tag{2.36}$$

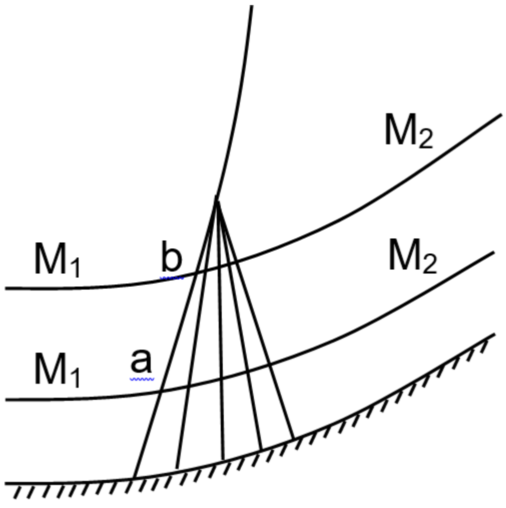


Fig. 2.7: Supersonic compression by turning

$$s_k - s_1 \infty n(\triangle\vartheta)^3 \quad (n\triangle\vartheta)(\triangle\vartheta)^2 \quad \vartheta(\triangle\vartheta)^2 \tag{2.37}$$

Thus, when the compression is achieved through a large number of weak shocks, the entropy increase can be reduced significantly compared to a single shock giving the same net deflection. It decreases as $\frac{1}{n^2}$. By contnuing the process of subdivision, the segments can be made vanishingly small ($\triangle\vartheta \to 0$), and in the limit, the smooth turn or isentropic compression is obtained.

When the shocks become vanishingly weak, they are almost straight Mach lines. Each segment of uniform flow becomes vanishingly narrow and finally coincides with a Mach line. Thus, the flow inclination and Mach number are constant on each Mach line. Thus, in the limit of smooth flow, the flow velocities and inclination are continuous, but their derivatives may still be discontinuous. The approximate expression for the change of speed across a very weak shock

$$\frac{\triangle w}{w} = -\frac{\vartheta}{\sqrt{M_1^2 - 1}} \tag{2.38}$$

becomes the differential equation

$$\frac{dw}{w} = -\frac{d\vartheta}{\sqrt{M_1^2 - 1}} \rightarrow \vartheta = \vartheta(M) \tag{2.39}$$

Due to the convergence of the Mach lines, the change form M_1 to M_2 on the streamline b occurs in a shorter distance than on the streamline a. Hence, the gradients of velocity and temperature on b are higher than those on a. An intersection of Mach lines would imply an infinitely high gradient for there would be two values of M at one point. However, this cannot occur since in the region where Mach lines converge and the gradients become very high the conditions are no longer isentropic. Before the Mach lines cross a shock wave is developed. Far from the corner, there would be a simple oblique shock corresponding to M_1

and ϑ. The convergence of Mach lines in a compression is a typical nonlinear effect: decreasing Mach number and increasing flow inclination both tend to make successive Mach lines steeper.

If a wall is placed along one of the streamlines, say b, where the gradients are still small enough for the flow to be isentropic; then an isentropic compression in a curved channel is obtained. Since this flow is isentropic, it may be reversed without violating the second law of thermodynamics.

2.4 Supersonic Expansion by Turning

Flow round a 'concave' turn, that is turns in which the wall is deflected in to the flow, undergoes compression through shock wave/Mach lines. Expansion takes place in a flow over a convex corner. In this case a turn through a single oblique wave is not possible.

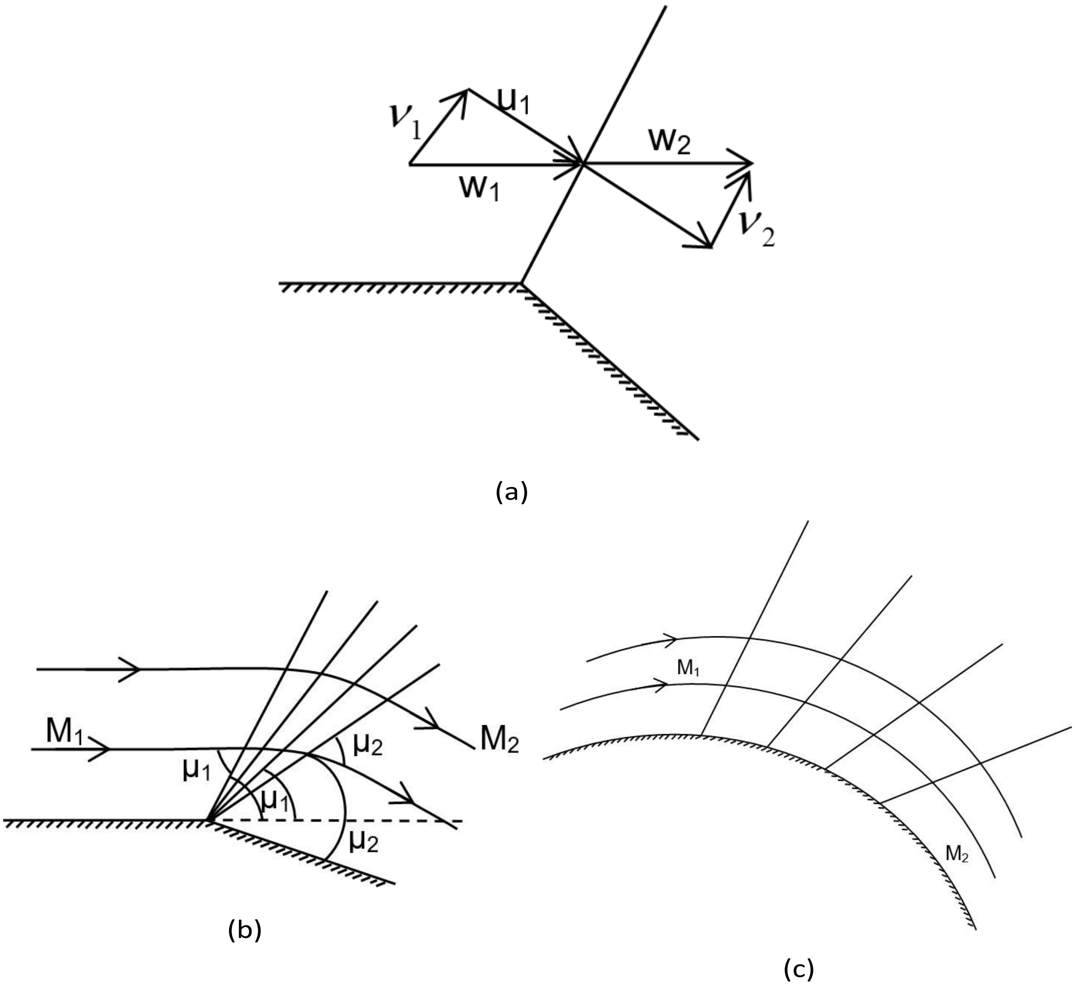


Fig. 2.8: Supersonic expansion by turning

Since $v_1 = v_2$, u_2 must be greater than u_1 $\rightarrow$ decrease in entropy. Hence expansion shocks are not possible.

The non-linear mechanism that steepens a compression produces the opposite effect in expansion. Instead of being convergent, the Mach lines are divergent.

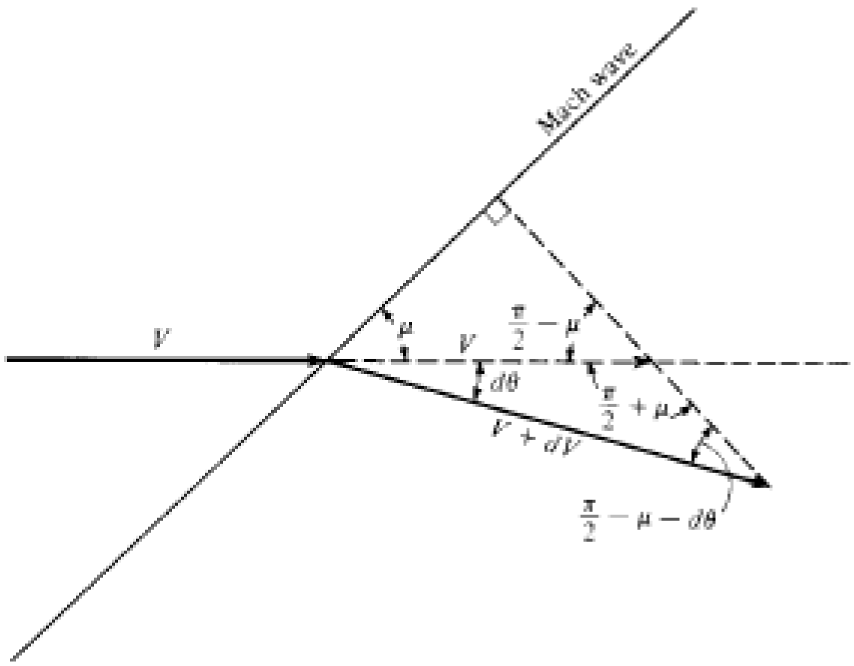


Fig. 2.9: Prandtl mayer expansion geometry

Consequently, there is a tendency to decrease gradients. Thus an expansion is isentropic throughout. The expansion at a corner occurs though a centered wave defined by a fan of straight Mach lines. The flow up to the corner is uniform at Mach number M_1 and thus the leading Mach wave must be straight at the Mach angle μ_1. The terminating Mach lines stands at the angle μ_2 (corresponding to M_2) to downstream wall. This centered wave is called a Prandtl-Meyer expansion fan.

From Fig. 2.9, for a given M_1, p_1, T_1 and ϑ_1 the M_2, p_2 and T_2 are needed to be calculated or predicted. The analysis can be started by considering the infinitesimal changes across a very weak wave produces by an infinitesimally small flow deflection, $d\vartheta$. From the law of sines,

$$\frac{V + dV}{V} = \frac{\sin(\pi/2 + \mu)}{\sin(\pi/2 - \mu - d\vartheta} \tag{2.40}$$

However, from trignometric identies,

$$\sin \frac{\pi}{2} + \mu = \sin \frac{\pi}{2} - \mu = \cos\mu \tag{2.41}$$

$$\sin \frac{\pi}{2} - \mu - d\vartheta = \cos(\mu + d\vartheta) = \cos\mu\cos d\vartheta - \sin\mu\sin d\vartheta \tag{2.42}$$

Solving above equations two equations into 2.40, we get

$$1 + \frac{dV}{V} = \frac{\cos\mu}{\cos\mu\cos d\vartheta - \sin\mu\sin d\vartheta} \tag{2.43}$$

For small $d\vartheta$, we can make the small angle assumptions $\sin d\vartheta \approx d\vartheta$ and $\cos d\vartheta \approx 1$. The the above equation becomes

$$1 + \frac{dV}{V} = \frac{\cos\mu}{\cos\mu - d\vartheta \quad \sin\mu} = \frac{1}{1 - d\vartheta \ \ tan\mu} \tag{2.44}$$

Recalling the series expansion (for $x < 1$),

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + ... \tag{2.45}$$

Equ. 2.44 can be expanded as (ignoring terms of second and higher order)

$$1 + \frac{dV}{V} = 1 + d\vartheta \quad \tan\mu + ... \tag{2.46}$$

Thus from the above equation,

$$d\vartheta = \frac{dV/V}{\tan\mu} \tag{2.47}$$

However, we know that mach angle is

$$\mu = \sin_{-1}\frac{1}{M} \tag{2.48}$$

which can be written as

$$\tan\mu = \frac{1}{\sqrt{M^2 - 1}} \tag{2.49}$$

Substituting the above equation into Equ. 2.47, we get

$$\boxed{d\vartheta = \sqrt{M^2 - 1}\frac{dV}{V}} \tag{2.50}$$

Equ. 2.50 is the governing differential equation for Prandtl-Meyer flow.

To analyze the entire Prandtl-Meyer expansion in Fig. 2.9, Equ. 2.50 must be integrated over the complete angle ϑ_2. Integrating Equ. 2.50 from regions 1 to 2,

$$\int_{\vartheta_1}^{\vartheta_2} d\vartheta = \int_{M_1}^{M_2} \sqrt{M^2 - 1}\frac{dV}{V} \tag{2.51}$$

The integral on the right hand side can be evaluated after dV/V is obtained in terms of M as ollows. From the definition of Mach number,

$$V = Ma$$

Hence,

$$ln\ V = ln\ M + ln\ a \tag{2.52}$$

Differentiating the above equation

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a} \tag{2.53}$$

For a calorically perfect gas the adiabatic energy equation can be written as

$$\left(\frac{a_o}{a}\right)^2 = \frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}M^2 \tag{2.54}$$

Solving for a,

$$a_o\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-1/2} \tag{2.55}$$

Differentiating the above equation,

$$\frac{da}{a} = -\left(\frac{\gamma - 1}{2}\right)M\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-1}dM \tag{2.56}$$

Substituting Equ. 2.56 into Equ. 2.53, we obtain

$$\frac{dV}{V} = \frac{1}{1 + \frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{2.57}$$

The above equation is desired relation for dV/V in terms of M, substitute it into Equ. 2.51,

$$\int_{\vartheta_1}^{\vartheta_2} d\vartheta = \vartheta_2 - 0 = \int_{M_1}^{M_2} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{2.58}$$

In the above equation, the integral

$$\nu(M) = \int \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2}M^2}\frac{dM}{M} \tag{2.59}$$

is called the Prandtl-Meyer function, and is given the symbol ν. Performing the integration, the above equation becomes

$$\boxed{\nu(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}}\tan^{-1}\sqrt{\frac{\gamma + 1}{\gamma - 1}(M^2 - 1)} - \tan^{-1}\sqrt{M^2 - 1}} \tag{2.60}$$

The constant of integration that would ordinarily appear in the above equation is not important, because it drops out when the Equ. 2.60 is substituted into Equ. 2.58. For convenience, it is chosen as zero such that $\nu(M) = 0$ when $M = 1$. Finally, we can now write Equ. 2.58 combined with Equ. 2.59, as

$$\boxed{\nu(M_2) - \nu(M_1)} \tag{2.61}$$

From Fig. 2.9, Equ. 2.61 and Equ. 2.60 allow the calculation of a Prandtl-Meyer expansion wave.

2.5 Simple and Non-simple regions

The isentropic compression and expansion waves are distinguished by straight Mach lines with constant conditions on each one and by the simple relation between flow deflection and Prandtl-Meyer function. A wave belongs to one of two families (+ or –), depending on whether the wall that produces it is to the left or

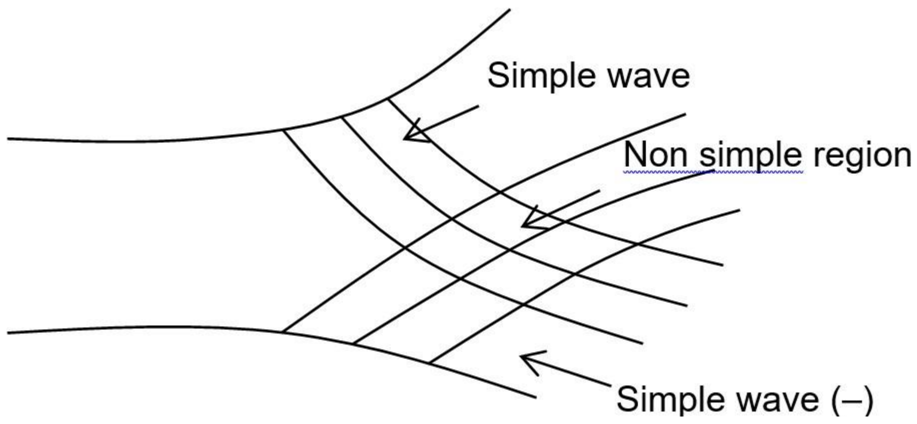


Fig. 2.10: Regions in isentropic supersonic flow

right of flow respectively. In the region where two simple waves of opposite family interact with each other, the flow is non-simple. The relation between v and ϑ is not the simple one given by $v = v \pm \vartheta$. These regions may be treated by the method of characteristics.

2.6 Regular reflection of oblique shocks from a solid boundary

Consider an oblique shock wave incident on a solid wall, as sketched in Fig. 4.18. Question: Does the shock wave disappear at the wall, or is it reflected downstream? If it is reflected, at what angle and what strength? The answer lies in the physical boundary condition at the wall, where the flow immediately adjacent to the wall must be parallel to the wall. In Figure 2.11 the flow in region 1 with Mach number M_1 is deflected through an angle ϑ at point A. This creates an oblique shock wave that impinges on the upper wall at point B. In region 2 behind this incident shock, the streamlines are inclined at an angle ϑ to the upper wall. All flow conditions in region 2 are uniquely defined by M_1 and ϑ through the oblique shock relations discussed earlier. At point B, in order for the flow to remain tangent to the upperwall, the streamlines in region 2 must be deflected downward through the angle ϑ. This can only be done by a second shock wave, originating at B, with sufficient strength to turn the flow through an angle 8, with an upstream Mach number of M_2. This second shock is called a rejected shock; its strength is uniquely defined by M_2 and ϑ, yielding the consequent properties in region 3. Because $M_2 < M_1$. the reflected shock wave is weaker than the incident shock, and the angle Φ it makes with the upper wall is not equal to β_1(i.e., the reflected shock wave is not specularly reflected).

2.7 Mach Reflection:

The appearance of subsonic regions in the flow complicates the problem. The complications are also encountered in shock reflections, when they are too strong to give the simple reflections. If M_2 after the incident shock is lower than the

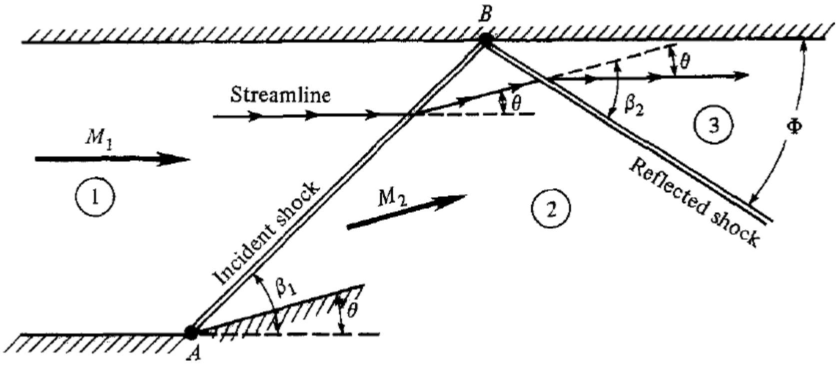


Fig. 2.11: Shock reflection

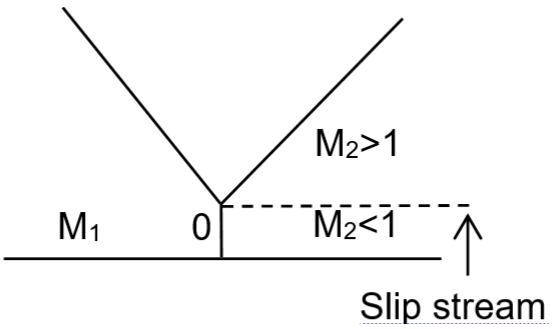


Fig. 2.12: Mach reflection

detachment Mach number for ϑ, then no solution with simple oblique wave is possible. A three-shock Mach reflection appears that satisfies the downstream conditions.

A normal, or, nearly normal, shock stem that appears near the wall forms a triple intersection point at O with the incident and reflected shocks. Due to the difference in entropy on streamlines above and below the triple point, the streamline that extends downstream from the triple point is a slipstream. The nearly normal shock is termed 'shock stem'.

The subsonic region behind the shock stem makes a local description of the configuration impossible. The triple point solution that occurs in a particular problem and the location of the triple point are determined by the downstream conditions which influence the subsonic part of the flow.

2.8 Shock-Expansion Theory

Oblique shock wave and simple isentropic wave relations can be used to analyze many 2-D supersonic flow problems, particularly for geometries with straight segments.

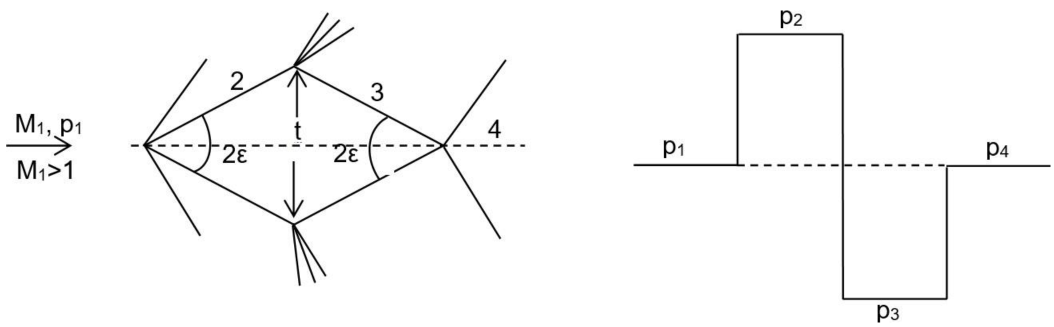


Fig. 2.13: Illustration of shock expansion theory for symmetrical diamond section

2.8.1 Diamond-section airfoil:

Consider a diamond section or double-wedge section airfoil with semi-vertex angle ϱ. Assume the semi-vertex angle to be sufficiently smaller than ϑ_{max} associated with the free stream Mach number M_1. An attached oblique shock appears at the nose that compresses the flow to pressure p_2

On the straight portion, downstream of the shock the flow remains uniform at M_2. The centered expansion at the shoulder expands the flow to pressure p_3 and the trailing edge shock recompresses it to nearly the free stream pressure ($p_4 \approx p_1$). Hence, an overpressure acts on the forward face and an under-pressure acts on the rearward face. Since the pressure on the two straight portions is unequal, a drag force acts on the airfoil. This drag force is given by

$$D = (p_2 - p_3)\cos\varrho . t \approx (p_2 - p_3)t \qquad perunitspan \tag{2.62}$$

Here t is the section thickness at the shoulder. Pressure values p_2 and p_3 can be obtained using the shock and expansion relations. This drag exists only in supersonic flow and is called ‘supersonic wave drag’.

2.8.2 Flat plat at incidence

Consider a flat plate of chord c set at an angle of attack . Due to no upstream influence, the streamlines ahead of the leading edge are straight and the upper surface flow is independent of lower surface. The flow on the upper surface turns at the nose through a centered expansion by the angle α whereas on the lower side the flow is turned through a compression angle α by an oblique shock. The reverse happens at the trailing edge.

From the uniform pressures on the two sides, the lift and drag forces are

$$L = (p_3 - p_2)\, c\cos\alpha \tag{2.63}$$

$$D = (p_3 - p_2)\, c\sin\alpha \tag{2.64}$$

The shock on the lower surface at the nose is weaker than the shock at the trailing edge on the upper surface (shock at higher Mach number). Hence, the increase in entropy for flow on the two sides is not same and consequently the streamline from the trailing edge is a slipstream inclined at a small angle relative

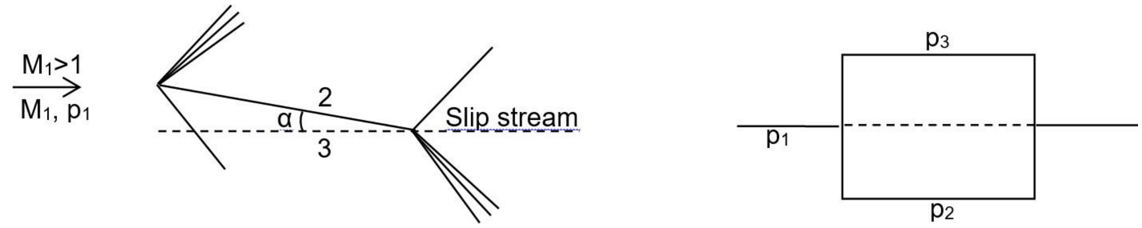


Fig. 2.14: Shock expansion theory flat plate

Fig. 2.15: Illustation of shock expansion theory on curved airfoil section

to the free stream.

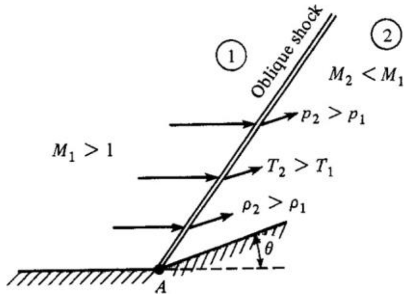
2.8.3 Curved airfoil section

An attached shock forms at the nose. Subsequently, continuous expansion occurs along the surface. The flow leaves at the trailing edge through an oblique shock. For the shocks to be attached, it is required that nose and tail be wedge shaped with half angle less than ϑ_{max}. Since the flow over the curved wall varies continuously, no simple expression for lift and drag forces is obtained in this case. If a larger portion of the flow field is considered, then the shocks and expansion waves will interact. The expansion fans attenuate the oblique shocks, making them weak and curved. At large distances they approach asymptotically the free-stream Mach lines. Due to the interaction the waves will reflect. The reflected wave system will alter the flow field. In shock-expansion theory, the reflected waves are neglected. For a diamond airfoil and a lifting flat plate, the reflected waves do not intercept the airfoil at all. Hence, the shock-expansion results are not affected.

Fig. 2.16: Atteneuation of wave by interaction around diamond section and flat plate

2.9 Numerical Problems

1. A uniform supersonic stream with M_1 = 3.0, p_1 = 1atm, and T_1 = 288 K encounters a compression corner which deflects the stream by an angle ϑ = 2.0. Calculate the shock wave angle, and p_2, T_2, M_2, $_{po2}$, and T_{o2}, behind the shock wave.



Solution: From the $\vartheta - \beta - M$ chart, for M_1 = 3 and ϑ = 20^0:

$$\boxed{\beta = 37.8^0}$$

Thus

$$M_{n1} = M_1 \sin\beta = 3\sin 37.8^0 = 1.839$$

From Table A.2, for M_{n1} = 1.839:

$$\frac{p_2}{p_1} = 3.783,\ \frac{T_2}{T_1} = 1.562,\ M_{n2} = 0.6078,\ and\ \frac{p_{o2}}{p_{o1}} = 0.7948.$$

Hence,

$$p_2 = \frac{p_2}{p_1}p_1 = 3.783 \times 1 = \boxed{3.783\ atm}$$

$$T_2 = \frac{T_2}{T_1}T_1 = 1.562 \times 288 = \boxed{449.9\ K}$$

$$M_2 = \frac{M_{n2}}{\sin(\beta - \vartheta)} = \frac{0.6078}{\sin 17.8^o} = \boxed{1.988}$$

From Table A.1, for $M_1 = 3$:

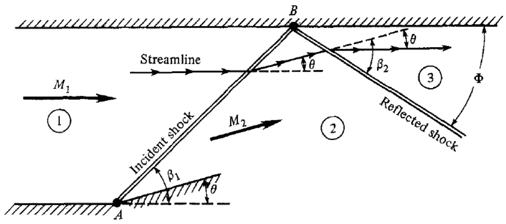
$$\frac{p_{o1}}{p_1} = 36.73 \; and \; \frac{T_{o1}}{T_1} = 2.8$$

Hence

$$p_{o2} = \frac{p_{o2}}{p_{o1}} \frac{p_{o1}}{p_1} p_1 = 0.7948 \times 36.73 \times 1 = \boxed{29.19 \; atm}$$

$$T_{o2} = T_{o1} = \frac{T_{o1}}{T_1} T_1 = 2.8 \times 288 = \boxed{806.4 \; K}$$

2. Consider a horizontal supersonic flow at Mach 2.8 with a static pressure and temperature of 1 atm and 519^oR, respectively. This flow passes over a compression corner with a defection angle of 16^o . The oblique shock generated at the corner propagates into the flow, and is incident on a horizontal wall. as shown in below figure. Calculate the angle Φ, made by the reflected shock wave with respect to the wall, and the Mach number, pressure, and temperature behind the reflected shock.



Solution: From the $\vartheta - \beta - M$ diagram, $\beta_1 = 35^o$

$$M_{n1} = M_1 \sin \beta_1 = 2.8 \sin 35 = 1.606$$

From Table A.2, for $M_{n1} = 1.606$:

$$\frac{p_2}{p_1} = 2.82, \frac{T_2}{T_1} = 1.388 \; and \; M_{n2} = 0.6684$$

Hence

$$M_2 = \frac{M_{n2}}{\sin(\beta_1 - \vartheta)} = \frac{0.6684}{\sin(35 - 16)} = 2.053$$

From the $\vartheta - \beta - M$ chart, for $M = 2.053$ and $\vartheta = 16^o$:

$$\beta_2 = 45.5^o$$

The component of the Mach number ahead of the reflected shock normal to the shock is M_{n2}, is given by

$$M_{n2} = M_2 \sin \beta_2 = 2.053 \times \sin 45.5 = 1.46$$

From Table A.2, for $M_{n2} = 1.46$:

$$\frac{p_3}{p_2} = 2.32, \quad \frac{T_3}{T_2} = 1.294, \text{ and } M_{n3} = 0.715$$

Where M_{n3} is the component of the Mach number behind the reflected shock normal to the shock. The Mach number in region 3 behind the reflected shock is given by

$$M_3 = \frac{M_{n3}}{\sin(\beta_2 - \vartheta)} = \frac{0.7157}{\sin(45.5 - 16)} = 1.45$$

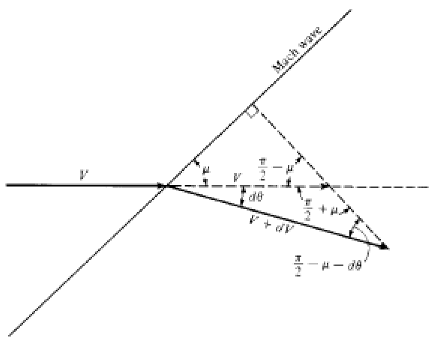
Also,

$$p_3 = \frac{p_3}{p_2}\frac{p_2}{p_1} p_1 = 2.32 \times 2.82 \times 1 = 6.54 \; atm$$

$$T_3 = \frac{T_3}{T_2}\frac{T_2}{T_1} T_1 = 1.294 \times 1.388 \times 519 = 932 \; R$$

$$\Phi = \beta_2 - \vartheta = 45.5 - 16 = 29.5$$

3. A uniform supersonic stream with $M_1 = 1.5$, $p_1 = 17001b/ft2$, and $T_1 = 460^o$ R encounters an expansion comer which deflects the stream by an angle $\vartheta 2 = 20^o$. Calculate M_2, p_2, T_2, p_{o2}, T_{o2}, and the angles the forward and rearward Mach lines make with respect to the upstream flow direction.



Solution:

From Table A.5, for $M_1 = 1.5$:

$$\nu_1 = 11.91^o \text{ and } \mu_1 = 41.81^o$$

So,

$$\nu_2 = \nu_1 + \vartheta_1 = 11.91 + 0 = 31.91^o$$

From Table A.5, for $\nu_2 = 31.91^o$:

$$M_2 = 2.207 \text{ and } \mu_2 = 26.95^o$$

From Table A.1, for $M_1 = 1.5$:

$$\frac{p_{o1}}{p_1} = 3.671 \text{ and } \frac{T_{o1}}{T_1} = 1.45$$

From Table A.1, for $M_2 = 2.207$:

$$\frac{p_{o2}}{p_2} = 10.81 \text{ and } \frac{T_{o2}}{T_2} = 1.974$$

The flow through an expansion wave is isentropic, hence $p_{o2} = p_{o1}$ and $T_{o2} = T_{o1}$. Thus,

$$p_2 = \frac{p_2}{p_{o2}}\frac{p_{o2}}{p_{o1}}\frac{p_{o1}}{p_1}p_1 = \frac{1}{10.81} \times 1 \times 3.671 \times 1700 = \boxed{577.3\ lb/ft^2}$$

$$T_2 = \frac{T_2}{T_{o2}}\frac{T_{o2}}{T_{o1}}\frac{T_{o1}}{T_1}T_1 = \frac{1}{1.975} \times 1 \times 1.45 \times 460 = \boxed{337.9^o\ R}$$

$$p_{o2} = p_{o1} = \frac{p_{o1}}{p_1}p_1 = 3.671 \times 1700 = \boxed{6241\ lb/ft^2}$$

$$T_{o2} = T_{o1} = \frac{T_{o1}}{T_1} = 1.45 \times 460 = \boxed{667^o\ R}$$

Finally,

$$\text{Angle of forward Mach Line} = \mu_1 = \boxed{41.81^o}$$

$$\text{Angle of rearward Mach Line} = \mu_2 - \vartheta_2 = 26.95 - 20 = \boxed{6.95^o}$$

4. Consider an infinitely thin flat plate at a 5^o angle of attack in a Mach 2.6 free stream. Calculate the lift and drag coefficients.

Solution: From Table A.5, for $M_1 = 2.6$:

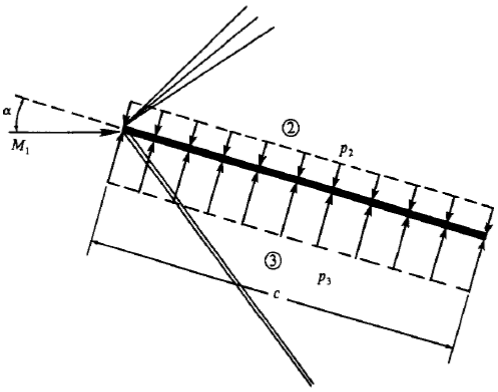
$$\nu_1 = 41.41^o$$

From the prandtl meyer function, Equ. 2.61,

$$\nu_2 = \nu_1 + \alpha = 41.41 + 5 = 46.41$$

From table A.5, for $\nu_2 = 46.41^o$:

$$M_2 = 2.85$$



From Table A.1, for $M_1 = 2.6$:

$$\frac{p_{o1}}{p_1} = 19.95$$

From Table A.1, for $M_2 = 2.85$:

$$\frac{p_{o2}}{p_2} = 29.29$$

Hence

$$\frac{p_2}{p_1} = \frac{p_2}{p_{o2}}\frac{p_{o2}}{p_{o1}}\frac{p_{o1}}{p_1} = \frac{1}{29.29} \times 1 \times 19.95 = 0.681$$

From the $\vartheta - \beta - M$ chart, for $M_1 = 2.6$ and $\vartheta = \alpha = 5^o$:

$$\beta = 26.5^o$$

Thus,

$$M_{n1} = M_1 \sin\beta = 2.6 \times \sin 26.5^o = 1.16$$

From Table A.2, for $M_{n1} = 1.16$:

$$\frac{p_3}{p_1} = 1.403.$$

The lift per unit span L' is

$$L' = (p_3 - p_2)\, c \cos\alpha$$

The drag per unit span D' is

$$D' = (p_3 - p_2)\, c \sin\alpha$$

Recalling, the dynamic pressure is given by $q = (\gamma/2)\, p_1 M_1^2$ we have

$$c_l = \frac{L'}{q_1 c} = \frac{2}{\gamma M_1^2}\left(\frac{p_3}{p_1} - \frac{p_2}{p_1}\right)\cos\alpha$$

$$c_l = \frac{2}{1.4 \times 2.6^2}(1.403 - 0.681)\cos 5^o = \boxed{0.152}$$

$$c_d = \frac{D'}{q_1 c} = \frac{2}{\gamma M_1^2}\left(\frac{p_3}{p_1} - \frac{p_2}{p_1}\right)\sin\alpha$$

$$c_d = \frac{2}{1.4 \times 2.6^2}(1.403 - 0.681)\sin 5^o = \boxed{0.0133}$$

Chapter 3

Subsonic compressible flow over airfoil

3.1 Irrotational flow

The vorticity is a point property of the flow, and is given by $\nabla \times V$. Vorticity is twice the angular velocity of a fluid element, $\nabla \times = 2\omega$. A flow where $\nabla \times V = 0$ throughout is called a rotaional flow. In contrast, a flow where $\nabla \times V = 0$ everywhere is called an irrotational flow.

Irrotational flows are usally simpler to analyze than rotational flows, the irrotationality condition $\nabla \times V = 0$ adds an extra simplification to the general equations of motion. Consider an irrotational flow in more detail. In cartesian coordinates, the mathematical statement of irrotational flow is

$$\nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\nabla \times V = i\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - j\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

For this equality to at every point in the flow,

$$\boxed{\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \qquad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}} \tag{3.1}$$

Equ. 3.1 are called the irrotationality conditions. Now considering the Euler's equations without body forces

$$\rho \frac{DV}{Dt} = -\nabla p$$

For steady flow, the x component of this equation is

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}$$

or

$$-\frac{\partial p}{\partial x} dx = \rho u \frac{\partial u}{\partial x} dx + \rho v \frac{\partial u}{\partial y} dx + \rho w \frac{\partial u}{\partial z} dx \tag{3.2}$$

From Equ. 3.1,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad and \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

Substituting the above relations into Equ. 3.2 we get

$$-\frac{\partial p}{\partial x}dx = \rho u\frac{\partial u}{\partial x}dx + \rho v\frac{\partial v}{\partial x}dx + \rho w\frac{\partial w}{\partial x}dx$$

or

$$-\frac{\partial p}{\partial x}dx = \frac{1}{2}\rho\frac{\partial u^2}{\partial x}dx + \frac{1}{2}\rho\frac{\partial v^2}{\partial x}dx + \frac{1}{2}\rho\frac{\partial w^2}{\partial x}dx \tag{3.3}$$

Similarly by considering the y and z components of Euler's equation,

$$-\frac{\partial p}{\partial y}dy = \frac{1}{2}\rho\frac{\partial u^2}{\partial y}dy + \frac{1}{2}\rho\frac{\partial v^2}{\partial y}dy + \frac{1}{2}\rho\frac{\partial w^2}{\partial y}dy \tag{3.4}$$

$$-\frac{\partial p}{\partial z}dz = \frac{1}{2}\rho\frac{\partial u^2}{\partial z}dz + \frac{1}{2}\rho\frac{\partial v^2}{\partial z}dz + \frac{1}{2}\rho\frac{\partial w^2}{\partial z}dz \tag{3.5}$$

Adding all the above three equations, we get

$$-\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz\right) = \frac{1}{2}\rho\frac{\partial V^2}{\partial x}dx + \frac{1}{2}\rho\frac{\partial V^2}{\partial y}dy + \frac{1}{2}\rho\frac{\partial V^2}{\partial z}dz \tag{3.6}$$

Where $V^2 = u^2 + v^2 + w^2$.

Equ. 3.6 is in the form of perfect differentials, and can be written as

$$-dp = \frac{1}{2}\rho d(V^2) \tag{3.7}$$

or

$$\boxed{dp = -\rho V\,dV} \tag{3.8}$$

3.2 Potential flow equation

The general conservation equations derived in previous chapter are simplified for the special case of irrotational flow. It allows the separate continuity, momentum and energy equations with the requisite dependent variables ρ, p, T, V etc to cascade into one governing equation with one dependent variable new defined as velocity potential.

For irrorational flow, $\nabla \times V = 0$. Hence, we can define a scalar function, $\Phi = \Phi(x, y, z)$, such that

$$\boxed{V = \nabla\Phi} \tag{3.9}$$

where Φ is called the velocity potential. In cartesian coordinates, since

$$V = ui + vj + wk$$

and

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}i + \frac{\partial\Phi}{\partial y}j + \frac{\partial\Phi}{\partial z}k$$

then, by comparision,

$$u = \frac{\partial\Phi}{\partial x} \qquad v = \frac{\partial\Phi}{\partial y} \qquad w = \frac{\partial\Phi}{\partial z} \tag{3.10}$$

Hence, if the velocity potential is known, the velocity can be obtained directly from the above equations.

As derived next, the velocity potential can be obtained from a single partial differential equation which physically describes an irrotational flow. In addition. we will assume steady, isentropic How. For simplicity, we will adopt subscript notation for derivatives of Φ as follows: $\partial\Phi/\partial x = \Phi_x$, $\partial\Phi/\partial y = \Phi_y$ and $\partial\Phi/\partial z = \Phi_z$.

The continuity equation for steady flow is

$$\nabla.(\rho V) = 0$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\rho\Phi_x + \frac{\partial}{\partial y}\rho\Phi_y + \frac{\partial}{\partial z}\rho\Phi_z = 0$$

$$\rho(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}) + \Phi_x\frac{\partial\rho}{\partial x} + \Phi_y\frac{\partial\rho}{\partial y} + \Phi_z\frac{\partial\rho}{\partial z} = 0 \tag{3.11}$$

Since, we are striving for an equation completely in terms of Φ, we eliminate ρ from Equ. 3.11 by using Euler's equation in the form of Equ. 3.8, which for an irrotational flow applies in any direction:

$$dp = -\rho V\,dV = -\frac{\rho}{2}d(V^2) = -\frac{\rho}{2}d(u^2 + v^2 + w^2)$$

$$dp = -\rho d\left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2}\right) \tag{3.12}$$

From the speed of sound, $a^2 = (\partial p/\partial\rho)_s$, Recalling the flow is isentropic, any flow change in pressure dp in the flow is followed by a corresponding change in density, $d\rho$. Hence,

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial\rho}\right)_s = a^2$$

$$d\rho = \frac{dp}{a^2} \tag{3.13}$$

Combining Equ. 3.12 and Equ. 3.13,

$$d\rho = -\frac{\rho}{a^2}\left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2}\right) \tag{3.14}$$

Considering changes in the x direction, the above equation yields,

$$\frac{\partial\rho}{\partial x} = -\frac{\rho}{a^2}\frac{\partial}{\partial x}\left(\frac{\Phi_x^2 + \Phi_y^2 + \Phi_z^2}{2}\right) \tag{3.15}$$

or

$$\frac{\partial\rho}{\partial x} = -\frac{\rho}{a^2}(\Phi_x\Phi_{xx} + \Phi_y\Phi_{yx} + \Phi_z\Phi_{zx}) \tag{3.16}$$

Similarly,

$$\frac{\partial \rho}{\partial y} = -\frac{\rho}{a^2}(\Phi_x \Phi_{xy} + \Phi_y \Phi_{yy} + \Phi_z \Phi_{zy}) \tag{3.17}$$

$$\frac{\partial \rho}{\partial z} = -\frac{\rho}{a^2}(\Phi_x \Phi_{xz} + \Phi_y \Phi_{yz} + \Phi_z \Phi_{zz}) \tag{3.18}$$

Substituting Equ. 3.16 through Equ. 3.18 into Equ. 3.11, canceling the ρ that appears in each term, and factoring out the second derivative of Φ, we get

$$\boxed{\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx} + \left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy} + \left(1-\frac{\Phi_z^2}{a^2}\right)\Phi_{zz} - \frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy} - \frac{2\Phi_x\Phi_z}{a^2}\Phi_{xz} - \frac{2\Phi_y\Phi_z}{a^2}\Phi_{yz} = 0} \tag{3.19}$$

Equ. 3.19 is called the velocity potential equation. Equ. 3.19 is not strictly in terms of Φ only, the variable speed of a sound a still appears. We need to express a in terms of Φ. From the energy equation,

$$h_o = constant$$

For a callorically perfect gas, the energy equation can be expressed as

$$c_p T + \frac{V^2}{2} = c_p T_o$$

$$\frac{\gamma R T}{\gamma - 1} + \frac{V^2}{2} = \frac{\gamma R T_o}{\gamma - 1}$$

$$\frac{a^2}{\gamma - 1} + \frac{V^2}{2} \quad u^2 + v^2 + w^2$$

$$\boxed{a^2 = a_o^2 - \frac{\gamma - 1}{2}\left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2\right)} \tag{3.20}$$

Since a_o is a known constant of the flow, the above equation gives the speed of sound a as a function of Φ.

Equ. 3.19 coupled with Equ. 3.20 represents a singlee equation for the unknown variable Φ. Equ. 3.20 represents a combination of continuity, momentum and energy equations. This leads to a general procedure for the solution of irrotational, isentropic flowfields:

- Solve for Φ from Equ. 3.19 and Equ. 3.20 for the specified boundary conditions of the given problem.
- Calculate u, v and w from Equ. 3.10, Hence $V = \sqrt{u^2 + v^2 + w^2}$.
- Calculate a from Equ. 3.20.
- Calculate $M = V/a$.
- Calculate T, p and ρ from Equ. 1.82, Equ. ?? and Equ. ?? respectively.

Hence, we see that once $\Phi = \Phi(x, y, z)$ is obtained, the *whole flowfield is known*. This shows the importance of Φ. The Equ. 3.19 combined with Equ. 3.20 is a non-linear partial differential equation. It applies to any irrotational, isentropic flow:

subsonic, transonic, supersonic or hypersonic. It also applies to incompressible flow, where $a \to \infty$, hence yielding the Laplace's equation

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0$$

3.3 Linearized velocity potential

Transport yourself back in time to the year 1940, and imagine that you are an aerodynamicist responsible for calculating the lift on the wing of a high-performance fighter plane. You recognize that the airspeed is high enough so that the wellestablished incompressible flow techniques of the day will give inaccurate results. Compressibility must be taken into account. However. you also recognize that the governing equations for compressible flow are nonlinear, and that no general solution exists for these equations. Numerical solutions are out of the question high-speed digital computers are still 15 years in the future. So, what do you do?The only practical recourse is to seek assumptions regarding the physics of theflow, which will allow the governing equations to become linear, but which at the same time do not totally compromise the accuracy of the real problem. In turn. these linear equations can be attacked by conventional mathematical techniques.

In this context, it is easy to appreciate why linear solutions to flow problems dominated the history of aerodynamics and gasdynamics up to the middle 1950s. In modern compressible flow, with the advent of the high-speed computer, the importance of linearized flow has been relaxed. Linearized solutions now take their proper role as closed-form analytic solutions useful for explicitly identifying trends and governing parameters, for highlighting some important physical aspects of the flow, and for providing practical formulas for the rapid estimation of aerodynamic forces and pressure distributions. In modern practice, whenever accuracy is desired the full nonlinear equations are solved numerically on a computer, as described in aubsequent chapters.

Consider a slender body immersed in a uniform flow. In the uniform flow, the velocity is V_∞ and is oriented in the x direction. In the perturbed flow, the local velocity is V , where $V = V_x i + V_y j + V_z k$, where V_x, V_y and V_z are now used to denote the x, y and z components of velocity, respectively. In this chapter u', v' and w' denote perturbations from the uniform flow, such that

$$V_x = V_\infty + u'$$

$$V_y = v'$$

$$V_z = w'$$

Here, u', v' and w' are the perturbation velocities in the x, y and z directions. In the perturbed flow, the pressure, density and temperature are p, ρ and T , respectively. In uniform stream, $V_x = V_\infty$, $V_y = 0$ and $V_z = 0$. Also in the unifrom stream, the pressure, density and temperature are p_∞, ρ_∞ and T_∞.

In terms of velocity potential

$$\nabla \Phi = V = (V_\infty + u')\, i + v' j + w' k$$

where Φ is now denoted as the "total velocity potential". The perturbation velocity potential φ is defined as

$$\frac{\partial \varphi}{\partial x} = u' \qquad \frac{\partial \varphi}{\partial y} = v' \qquad \frac{\partial \varphi}{\partial z} = w'$$

Then,

$$\Phi(x, y, z) = V_\infty x + \varphi(x, y, z)$$

where

$$V_x = V_\infty + u' = \frac{\partial \Phi}{\partial x} = V_\infty + \frac{\partial \varphi}{\partial x}$$
$$V_y = v' = \frac{\partial \Phi}{\partial y} = \frac{\partial \varphi}{\partial y}$$
$$V_z = w' = \frac{\partial \Phi}{\partial z} = \frac{\partial \varphi}{\partial z}$$

Also,

$$\Phi_{xx} = \frac{\partial^2 \varphi}{\partial x^2}$$
$$\Phi_{yy} = \frac{\partial^2 \varphi}{\partial y^2}$$
$$\Phi_{zz} = \frac{\partial^2 \varphi}{\partial z^2}$$

Considering the velocity potential Equ. 3.19. Multiplying it by a^2 and sustituting $\Phi = V_\infty x + \varphi$, we get

$$\left[a^2 - \left(V_\infty + \frac{\partial \varphi}{\partial x}\right)^2\right] \frac{\partial^2 \varphi}{\partial x^2} + \left[a^2 - \left(\frac{\partial \varphi}{\partial y}\right)^2\right] \frac{\partial^2 \varphi}{\partial y^2} + \left[a^2 - \left(\frac{\partial \varphi}{\partial z}\right)^2\right] \frac{\partial^2 \varphi}{\partial z^2}$$
$$-2\left(V_\infty + \frac{\partial \varphi}{\partial x}\right) \frac{\partial \varphi}{\partial y} \frac{\partial^2 \varphi}{\partial x \, \partial y} - 2\left(V_\infty + \frac{\partial \varphi}{\partial x}\right) \frac{\partial \varphi}{\partial z} \frac{\partial^2 \varphi}{\partial x \, \partial z} - 2 \frac{\partial \varphi}{\partial y} \frac{\partial \varphi}{\partial z} \frac{\partial^2 \varphi}{\partial y \, \partial z} = 0 \tag{3.21}$$

The above equation is called the perturbation-velocity potential equation. Recasting the above equation in terms of velocities

$$\begin{aligned} &\left[a^2 - (V_\infty + u')^2\right] \frac{\partial u'}{\partial x} + [a^2 - v'^2] \frac{\partial v'}{\partial y} + [a^2 - w'^2] \frac{\partial w'}{\partial z} \\ &-2(V_\infty + u')v' \frac{\partial u'}{\partial y} - 2(V_\infty + u')w' \frac{\partial u'}{\partial z} - 2v'w' \frac{\partial v'}{\partial z} = 0 \end{aligned} \tag{3.22}$$

Since the total enthalpy is constant throughout the flow,

$$h_\infty + \frac{V_\infty^2}{2} = h + \frac{V^2}{2} = h + \frac{(V_\infty + u')^2 + v'^2 + w'^2}{2}$$

or

$$\frac{a_\infty^2}{\gamma - 1} + \frac{V_\infty^2}{2} = \frac{a^2}{\gamma - 1} + \frac{(V_\infty + u')^2 + v'^2 + w'^2}{2}$$

$$a^2 = a_\infty^2 - \frac{\gamma - 1}{2}(2u'V_\infty + u'^2 + v'^2 + w'^2) \tag{3.23}$$

Substituting Equ. 3.23 into Equ. 3.22 and rearranging,

$$
\begin{aligned}
&(1 - M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \\
&= M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty} + \frac{\gamma+1}{2}\frac{u'^2}{V_\infty^2} + \frac{\gamma-1}{2}\left(\frac{v'^2 + w'^2}{V_\infty^2}\right)\right]\frac{\partial u'}{\partial x} \\
&+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty} + \frac{\gamma+1}{2}\frac{v'^2}{V_\infty^2} + \frac{\gamma-1}{2}\left(\frac{w'^2 + u'^2}{V_\infty^2}\right)\right]\frac{\partial v'}{\partial y} \\
&+M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty} + \frac{\gamma+1}{2}\frac{w'^2}{V_\infty^2} + \frac{\gamma-1}{2}\left(\frac{u'^2 + v'^2}{V_\infty^2}\right)\right]\frac{\partial w'}{\partial z} \\
&+M_\infty^2\left[\frac{v'}{V_\infty}\left(1 + \frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) + \frac{w'}{V_\infty}\left(1 + \frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right) + \frac{u'w'}{V_\infty^2}\left(\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}\right)\right]
\end{aligned}
\tag{3.24}
$$

Equ. 3.24 is still an exact equation for irrotational, isentropic flow. It is simply an expanded form of the perturbation-velocity potential equation. Note that the left-hand side of Equ. 3.24 is linear, but the right-hand side is not. Also recall that we have not said anything about the size of the perturbation velocities u', v' and w'. Now, assume u', v' and w' are small compared to V_∞:

$$\frac{u'}{V_\infty}, \frac{v'}{V_\infty} and \frac{w'}{V_\infty} << 1 \qquad \left(\frac{u'}{V_\infty}\right)^2, \left(\frac{v'}{V_\infty}\right)^2 and \left(\frac{w'}{V_\infty}\right)^2 <<< 1$$

- For $0 \leq M_\infty \leq 0.8$ and $M_\infty \geq 1.2$, the magnitude of

  $$M_\infty^2\left[(\gamma+1)\frac{u'}{V_\infty} + ...\right]\frac{\partial u'}{\partial x}$$

  is small in comparision to the magnitude of

  $$(1 - M_\infty^2)\frac{\partial u'}{\partial x}$$

  Thus ignore the former term

- For $M_\infty \leq 5$ (approx),

  $$M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty} + ...\right]\frac{\partial v'}{\partial y}$$

  is small in comparision to $\partial v'/\partial y$,

  $$M_\infty^2\left[(\gamma-1)\frac{u'}{V_\infty} + ...\right]\frac{\partial w'}{\partial z}$$

  is small in comparision to $\partial w'/\partial z$, and

  $$M_\infty^2\left[\frac{v'}{V_\infty}\left(1 + \frac{u'}{V_\infty}\right)\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) + ...\right] \approx 0$$

Thus, ignore these terms in comparision to the left hand side of Equ. 3.24, with these order-of-magnitude comparisions, Equ. 3.24 reduces to

$$(1 - M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \tag{3.25}$$

or in terms of the perturbation velocity potential,

$$\boxed{(1 - M_\infty^2)\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}} \tag{3.26}$$

The Equ. 3.26 is the Linearized velocity potential equation.

3.4 Linearized pressure Coefficient

The pressure co-efficient c_p is defined as

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}$$

where p is the local pressure, and p_∞, ρ_∞ and V_∞ are the pressure, density and velocity respectively. An alternative form of the pressure coefficient, convenient for compressible flow, can be obtained as follows:

$$\frac{1}{2}\rho_\infty V_\infty^2 = \frac{1}{2}\frac{\gamma p_\infty}{\gamma p_\infty}\rho_\infty V_\infty^2 = \frac{\gamma}{2}p_\infty \frac{V_\infty^2}{a_\infty^2} = \frac{\gamma}{2}p_\infty M_\infty^2 \tag{3.27}$$

Hence Equ. 3.4 becomes

$$C_p = \frac{p - p_\infty}{(\gamma/2)p_\infty M_\infty^2} = \frac{p_\infty(p/p_\infty - 1)}{(\gamma/2)p_\infty M_\infty^2} \tag{3.28}$$

Hence,

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)} \tag{3.29}$$

We now proceed to obtain approx expression for C_p that is constant with linearized theory. Since the total enthalpy is constant,

$$h + \frac{V^2}{2} = h_\infty + \frac{V_\infty^2}{2}$$

For a calorically perfect gas, the above equation becomes

$$T + \frac{V^2}{2c_p} = T_\infty + \frac{V_\infty^2}{2c_p} \tag{3.30}$$

$$T - T_\infty = \frac{V_\infty^2 - V^2}{2c_p} = \frac{V_\infty^2 - V^2}{2\gamma R/(\gamma - 1)} \tag{3.31}$$

$$\frac{T}{T_\infty} - 1 = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{\gamma R T_\infty} = \frac{\gamma - 1}{2}\frac{V_\infty^2 - V^2}{a_\infty^2} \tag{3.32}$$

Since

$$V^2 = (V_\infty + u')^2 + v'^2 + w'^2$$

Equ. 3.32 becomes

$$\frac{T}{T_\infty} = 1 - \frac{\gamma - 1}{2a_\infty^2}(2u'V_\infty + u'^2 + v'^2 + w'^2) \tag{3.33}$$

Since the flow is isentropic, $p/p_\infty = (T/T_\infty)^{\gamma/(\gamma-1)}$, and the above equation becomes

$$\frac{p}{p_\infty} = \left[1 - \frac{\gamma - 1}{2a_\infty^2}\left(2u'V_\infty + u'^2 + v'^2 + w'^2\right)\right]^{\gamma/(\gamma-1)} \tag{3.34}$$

or

$$\frac{p}{p_\infty} = \left[1 - \frac{\gamma - 1}{2}M_\infty^2\left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2}\right)\right]^{\gamma/(\gamma-1)} \tag{3.35}$$

Equ. 3.35 is an exact expression. However, considering small perturbations: $u'/V_\infty \ll 1$: u'^2/V_∞^2, and $w'^2/V_\infty^2 \lll 1$. Hence the above equation is of the form

$$\frac{p}{p_\infty} = (1 - \varrho)^{\gamma/(\gamma-1)}$$

where ϱ is small. Hence from the binomial expresion, neglecting higher order terms,

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{\gamma - 1}\varrho + \ldots\ldots\ldots\ldots\ldots \tag{3.36}$$

Thus Equ. 3.35 can be expressed in the above form of equation and neglecting higher order terms:

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{2}M_\infty^2\left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2}\right) + \ldots\ldots\ldots\ldots \tag{3.37}$$

Substitute the aboveequation into Equ. 3.4 gives

$$C_p = \frac{2}{\gamma M_\infty^2}\left[1 - \frac{\gamma}{2}M_\infty^2\left(\frac{2u'}{V_\infty} + \frac{u'^2 + v'^2 + w'^2}{V_\infty^2}\right) + \ldots\ldots\ldots\ldots - 1\right] \tag{3.38}$$

$$C_p = -\frac{2u'}{V_\infty} - \frac{u'^2 + v'^2 + w'^2}{V_\infty^2} + \ldots\ldots\ldots\ldots\ldots \tag{3.39}$$

Negelecting small terms

$$\boxed{C_p = -\frac{2u'}{V_\infty}} \tag{3.40}$$

3.5 Prandtl-Glauert compressibility corrections

Consider the compressible subsonic flow over a thin airfoil at small angle of attack (hence small perturbations). The usual inviscid flow boundary condition must hold at the surface, i.e., the flow velocity must be tangent to the surface.

$$\frac{df}{dx} = \frac{v}{V_\infty + u} = \tan\vartheta$$

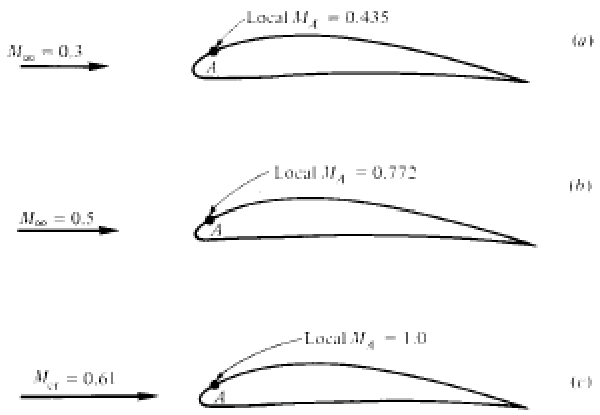


Fig. 3.1: Definition of critical Mach number. Point A is the location of minimum pressure on the top surface of the airfoil.

$$\boxed{C_p = \frac{C_{p_o}}{\sqrt{1 - M_\infty^2}}} \tag{3.41}$$

$$\boxed{C_L = \frac{C_{L_o}}{\sqrt{1 - M_\infty^2}}} \tag{3.42}$$

$$\boxed{C_M = \frac{C_{M_o}}{\sqrt{1 - M_\infty^2}}} \tag{3.43}$$

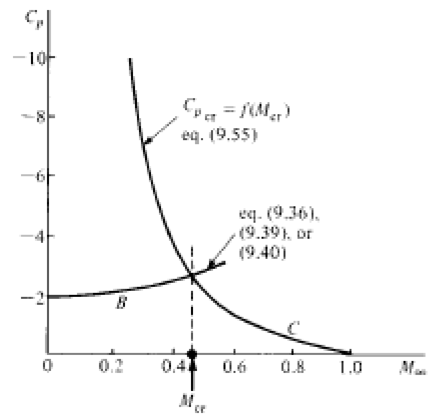
3.6 Critical Mach number

At high-subsonic flight speeds, the local speed of the airflow can reach the speed of sound where the flow accelerates around the aircraft body and wings. The speed at which this development occurs varies from aircraft to aircraft and is known as the critical Mach number. The resulting shock waves formed at these points of sonic flow can greatly reduce power, which is experienced by the aircraft as a sudden and very powerful drag, called wave drag. To reduce the number and power of these shock waves, an aerodynamic shape should change in cross sectional area as smoothly as possible.

The critical Mach number can be evaluated as follows

$$\frac{p_A}{p_B} = \left(\frac{1 + \frac{\gamma-1}{2}M_\infty^2}{1 + \frac{\gamma-1}{2}M_A^2}\right)^{\gamma/(\gamma-1)} \tag{3.44}$$

$$C_{pA} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1 + \frac{\gamma-1}{2}M_\infty^2}{1 + \frac{\gamma-1}{2}M_A^2}\right)^{\gamma/(\gamma-1)} - 1\right] \tag{3.45}$$



Setting $M_A = 1$, $M_\infty = M_{cr}$, and $C_p = C_{p_{cr}}$, we get

$$\boxed{C_{p_{cr}} = \frac{2}{\gamma M_{cr}^2}\left[\left(\frac{1+\frac{\gamma-1}{2}M_{cr}}{1+\frac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)} - 1\right]} \tag{3.46}$$

3.7 Drag divergence Mach number

The drag divergence Mach number (not to be confused with critical Mach number) is the Mach number at which the aerodynamic drag on an airfoil or airframe begins to increase rapidly as the Mach number continues to increase. This increase can cause the drag coefficient to rise to more than ten times its low speed value.

The value of the drag divergence Mach number is typically greater than 0.6; therefore it is a transonic effect. The drag divergence Mach number is usually close to, and always greater than, the critical Mach number. Generally, the drag coefficient peaks at Mach 1.0 and begins to decrease again after the transition into the supersonic regime above approximately Mach 1.2.

The large increase in drag is caused by the formation of a shock wave on the upper surface of the airfoil, which can induce flow separation and adverse pressure gradients on the aft portion of the wing. This effect requires that aircraft intended to fly at supersonic speeds have a large amount of thrust. In early development of transonic and supersonic aircraft, a steep dive was often used to provide extra acceleration through the high drag region around Mach 1.0. This steep increase in drag gave rise to the popular false notion of an unbreakable sound barrier, because it seemed that no aircraft technology in the foreseeable future would have enough propulsive force or control authority to overcome it. Indeed, one of the popular analytical methods for calculating drag at high speeds, the Prandtl-Glauert rule, predicts an infinite amount of drag at Mach 1.0. Two of the important technological advancements that arose out of attempts to conquer the sound barrier were the Whitcomb area rule and the supercritical airfoil. A supercritical airfoil is shaped specifically to make the drag divergence Mach number as high as possible, allowing aircraft to fly with relatively lower drag at high subsonic and low transonic speeds.

3.8 Area rule

The Whitcomb area rule, also called the transonic area rule, is a design technique used to reduce an aircraft's drag at transonic and supersonic speeds, particularly between Mach 0.75 and 1.2. The area rule says that two airplanes with the same longitudinal cross-sectional area distribution have the same wave drag, independent of how the area is distributed laterally (i.e. in the fuselage or in the wing). Furthermore, to avoid the formation of strong shock waves, this total area distribution must be smooth. As a result, aircraft have to be carefully arranged so that at the location of the wing, the fuselage is narrowed or "waisted", so that the total area does not change much. Similar but less pronounced fuselage waisting is used at the location of a bubble canopy and perhaps the tail surfaces.

3.9 Supercritical airfoil

A supercritical airfoil is an airfoil designed, primarily, to delay the onset of wave drag in the transonic speed range. Supercritical airfoils are characterized by their flattened upper surface, highly cambered ("downward-curved") aft section, and larger leading edge radius compared with NACA 6-series laminar airfoil shapes. Standard wing shapes are designed to create lower pressure over the top of the wing. The camber of the wing determines how much the air accelerates around the wing. As the speed of the aircraft approaches the speed of sound the air accelerating around the wing will reach Mach 1 and shockwaves will begin to form. The formation of these shockwaves causes wave drag. Supercritical airfoils are designed to minimize this effect by flattening the upper surface of the wing.

3.10 Numerical Problems

1. The low-speed lift coefficient for an NACA 2412 airfoil at an angle of attack of 4^0 is 0.65. Using the Prandtl-Glauert rule, calculate the lift coefficient for $M_\infty = 0.7$.

   Solution: Given

$$\alpha = 4^o$$

$$M_\infty = 0.65$$

   We know,

$$c_l = \frac{c_{l,0}}{\sqrt{1 - M_\infty^2}} = \frac{0.65}{\sqrt{1 - 0.7^2}} = 1.275$$

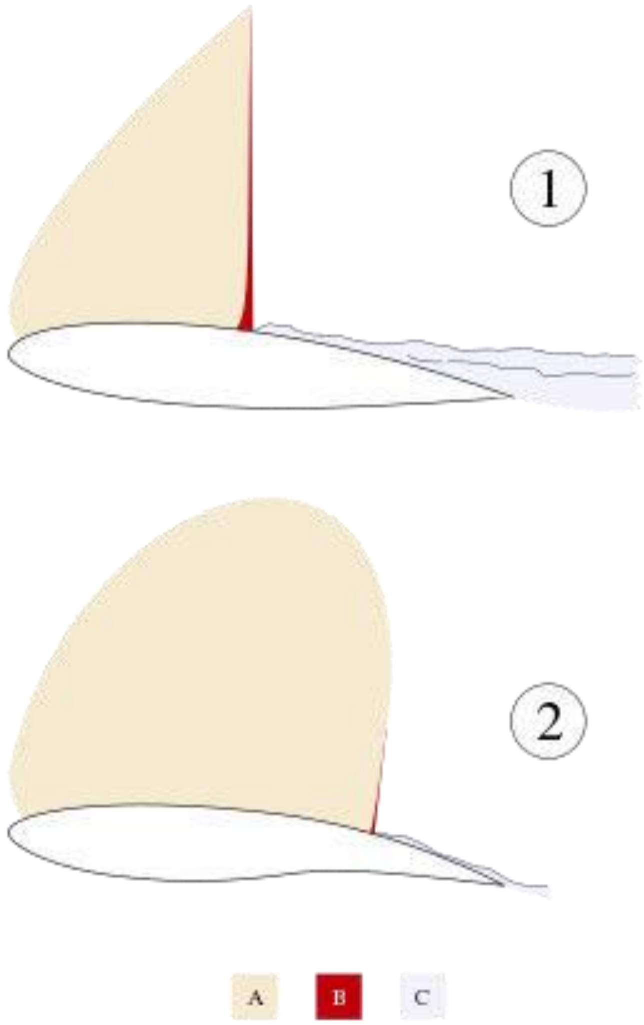


Fig. 3.2: Conventional (1) and supercritical (2) airfoils at identical free stream Mach number. Illustrated are: A, Supersonic flow region; B, Shock wave; C, Area of separated flow. The supersonic flow over a supercritical airfoil terminates in a weaker shock, thereby postponing shock-induced boundary layer separation.

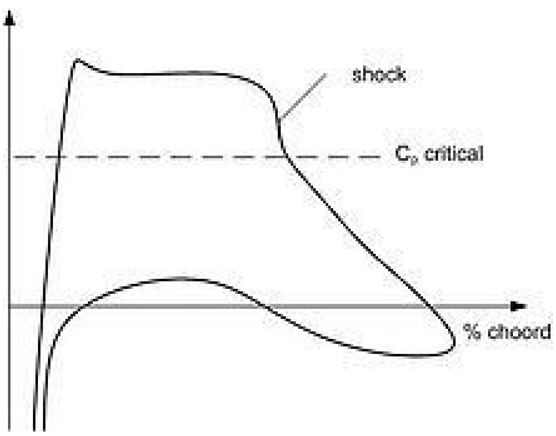


Fig. 3.3: Supercritical airfoil Mach Number/pressure coefficient diagram. The sudden increase in pressure coefficient at midchord is due to the shock. (y-axis:Mach number (or pressure coefficient, negative up); x-axis: position along chord, leading edge left)

Chapter 4
Linearized Supersonic Flows and Hypersonic Flows

4.1 Linearized supersonic pressure co-efficient

The linearized perturbation-velocity potential equation for two dimensional flows derived in the previous chapter is repeated here which is of the form

$$\beta^2\varphi_{xx} + \varphi_{yy} = 0 \qquad (4.1)$$

For subsonic flow, where $\beta = \sqrt{1 - M_\infty^2}$, and the form of

$$\lambda^2\varphi_{xx} - \varphi_{yy} = 0 \qquad (4.2)$$

for supersonic flow, where $\lambda = \sqrt{M_\infty^2 - 1}$. The difference between Eqs 4.1 and 4.2 is fundamental, for they are elliptic and hyperbolic partial differential equations, respectively.

Consider the supersonic flow over a body or surface which introduces small changes in the flowfield, i.e., flow over a thin airfoil, over a mildly wavy wall, or over a small hump in a surface as sketched in Figure 4.1.

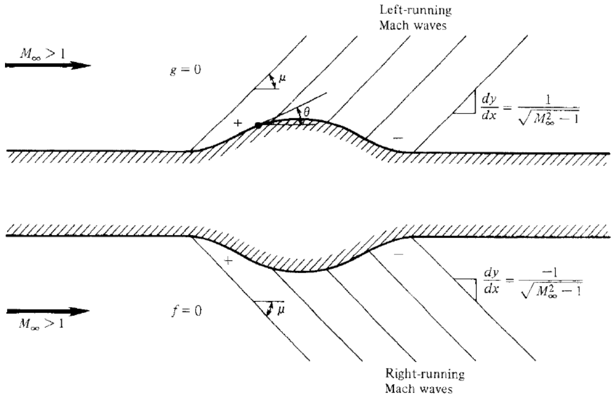


Fig. 4.1: Linearized supersonic flow over a bump

The Equ. 4.2, which governs the flow is of the form of the classical wave equation. Its general solution is

$$\varphi = f(x - \lambda y) + g(x + \lambda y) \qquad (4.3)$$

which can be verified by direct substitution into Equ. 4.2. Examining the particular solution where g =0, and hence $\varphi = f(x - \lambda y)$, we see that lines of constant φ correspond to $x - \lambda y = const$, or

$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}} \tag{4.4}$$

Recalling that the Mach ang;e $\mu = arcsin\ (1/M_\infty)$, we see that lines of constant φ are the family of left-running Mach lines as sketched in upper half of Figure 4.1. In turn if $f = 0$ in Equ.4.1, then lines of constant φ are the family of right-running Mach lines shown in lower half of Figure. 4.1.

Returning to Equ. 4.2, letting $g = 0$, we have

$$\varphi = f\,(x - \lambda y) \tag{4.5}$$

Hence,

$$u' = \frac{\partial \varphi}{\partial x} = f' \tag{4.6}$$

and

$$v' = \frac{\partial \varphi}{\partial y} = -\lambda f' \tag{4.7}$$

where f' reprents the derivative with respect to the argument, $(x - \lambda y)$. Combing the above two equations, we get

$$u' = -\frac{v'}{\lambda} \tag{4.8}$$

The slope of the left running Mach waves can be calculated as

$$\tan\vartheta = \frac{dy}{dx} = \frac{v'}{V_\infty + u'} \tag{4.9}$$

For small perturbations, $u' << V_\infty$ and $\tan\vartheta \approx \vartheta$. Hence, the above equation becomes

$$v' = V_\infty \vartheta \tag{4.10}$$

Substituting Equ. 4.10 into 4.8,

$$u' = -\frac{V_\infty \vartheta}{\lambda} \tag{4.11}$$

Recalling subsonic linearized pressure coefficient

$$C_p = -\frac{2u'}{V_\infty} \tag{4.12}$$

Therefore from Equs 4.12 and 4.11, the pressure coefficient on the surface is

$$C_p = -\frac{2u'}{V_\infty} = \frac{2\vartheta}{\infty} \tag{4.13}$$

or

$$\boxed{C_p = -\frac{2\vartheta}{\sqrt{M_\infty^2 - 1}}} \tag{4.14}$$

Equation 4.14 is an important result. It is the linearized supersonic surface pressure coefficient, and it states that C_p is directly proportional to the local surface inclination with respect to the free stream. It holds for any slender two-dimensional shape. For example, consider the biconvex airfoil shown in Fig. 4.2. At two arbitrary points A and B on the top surface,

$$C_{pA} = \frac{2\vartheta_A}{\sqrt{M_\infty^2 - 1}} \quad and \quad C_{pB} = -\frac{2\vartheta_B}{\sqrt{M_\infty^2 - 1}} \tag{4.15}$$

respectively.

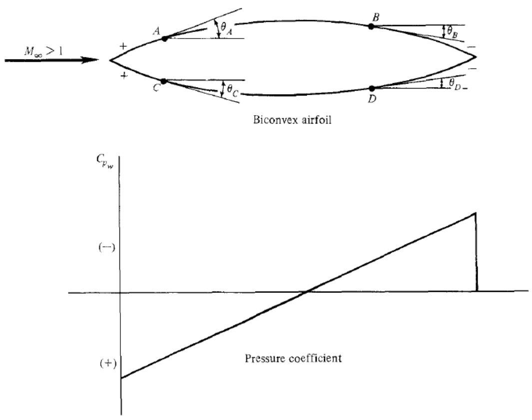


Fig. 4.2: Schematic of the lineari edp ressure coefficient over a biconvex airfoil

The contrast between subsonic and supersonic flows can be seen by comparing Equs. 4.12 and 4.14. In subsonic flow, Equ. 4.12 shows that C_p increases when M_∞ increases. However, for supersonic flow, Equ. 4.14 shows that C_p decreases when

4.2 Hypersonic Flows

In general words the flows with Mach number higher than 5 are categorized under hypersonic flows. However, hypersonic flow is best defined as that regime where certain physical flow phenornena become progressively more important as the Mach number is increased to higher values. In some cases, one or more of these phenomena may become important above Mach 3, whereas in other cases they may not be compelling until Mach 7 or higher.

4.2.1 Qualitative aspects of Hypersonic flows

The qualities of hypersonic flows can be listed as below:

1. Thin Shock layers

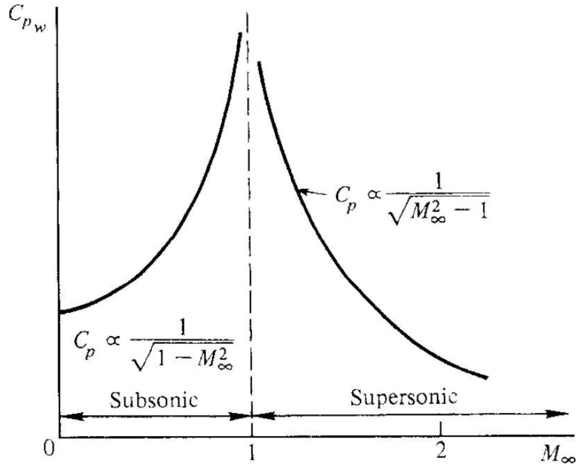


Fig. 4.3: Variation of the linearized preswre coefficient with Mach number.

2. Entropy layer

3. Viscous interaction

4. High-Temperature flows

4.2.1.1 Thin shock layer

For flow over a hypersonic body, the distance between the body and the shock wave is very small. The flowfield between the shock wave and the body is defined as the shock layer, and for hypersonic speeds this shock layer is usually quite thin. For example, consider the Mach 36 flow of a calorically perfect gas with a ratio of specific heats, $\gamma = c_p/c_v = 1.4$, over a wedge of 15^o half-angle. From standard oblique shock theory the shock wave angle will be only 18^o as shown in Figure. 15.3. If high-temperature, chemically reacting effects are included, the shock wave angle will be even smaller. Clearly, this shock layer is thin. It is a basic characteristic of hypersonic flows.

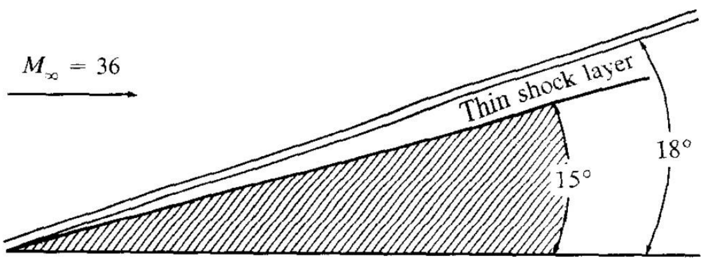


Fig. 4.4: Illustration of a thin shock layer at hypersonic Mach numbers

4.2.1.2 Entropy Layer

Consider the wedge shown in Figure. 4.4, except now with a blunt nose, as sketched in Figure. 4.5. 15.4. At hypersonic Mach numbers, the shock layer over the blunt nose is also very thin, with a small shock detachment distance d. In the nose region, the shock wave is highly curved. Recall that the entropy of the flow increases across a shock wave, and the stronger the shock, the larger the entropy increase. A streamline passing through the strong, nearly normal portion of the curved shock near the centerline of the flow will experience a larger entropy increase than a neighboring streamline which passes through a weaker portion of the shock further away from the centerline. Hence, there are strong entropy gradients generated in the nose region; this "entropy layer" flows downstream, and essentially wets the body for large distances from the nose, as shown in Fig. 4.5. The boundary layer along the surface grows inside this entropy layer, and is affected by it.

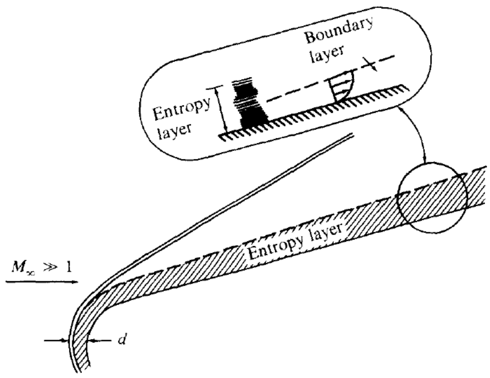


Fig. 4.5: Illustration of the entropy layer of a blunt-nosed slender body at hypersonic speeds

4.2.1.3 Viscous Interaction

Consider a boundary layer on a flat plate in a hypersonic flow, as sketched in Fig. 15.5. A high-velocity, hypersonic flow contains a large amount of kinetic energy; when this flow is slowed by viscous effects within the boundary layer, the lost kinetic energy is transformed (in part) into internal energy of the gas-this is called viscous dissipation. In turn, the temperature increases within the boundary layer; a typical temperature profile within the boundary layer is also sketched in Figure. 4.6. The characteristics of hypersonic boundary layers are dominated by such temperature increases. For example, the viscosity coefficient increases with temperature, and this by itself will make the boundary layer thicker. In addition, because the pressure p is constant in the normal direction through a boundary layer, the increase in temperature T results in a decrease in density ρ through the equation of state $p = \rho/RT$. In order to pass the required mass flow through the boundary layer at reduced density, the boundary layer thickness must be larger. Both of these phenomena combine to make hypersonic boundary layers grow more rapidly than at slower speeds. Indeed, the flat plate compressible laminar boundary layer thickness δ grows essentially as

$$\delta \propto \frac{M_\infty^2}{\sqrt{Re_x}} \tag{4.16}$$

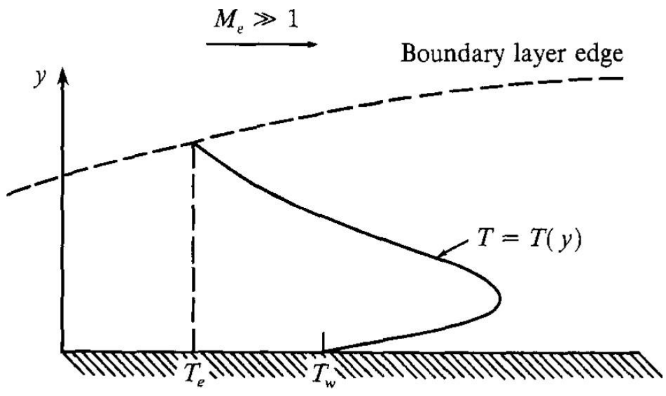


Fig. 4.6: Schematic of a temperature profile in a hypersonic boundary layer

4.2.1.4 High-Temperature Flows

As discussed earlier, the kinetic energy of a high-speed, hypersonic flow is dissipated by the influence of friction within a boundary layer. The extreme viscous dissipation that occurs within hypersonic boundary layers can create very high temperatures-high enough to excite vibrational energy internally within molecules, and to cause dissociation and even ionization within the gas. If the surface of a hypersonic vehicle is protected by an ablative heat shield, the products of ablation are also present in the boundary layer, giving rise to complex hydrocarbon chemical reactions. On both accounts, we see that the surface of a hypersonic vehicle can be wetted by a chemically reacting boundary layer.

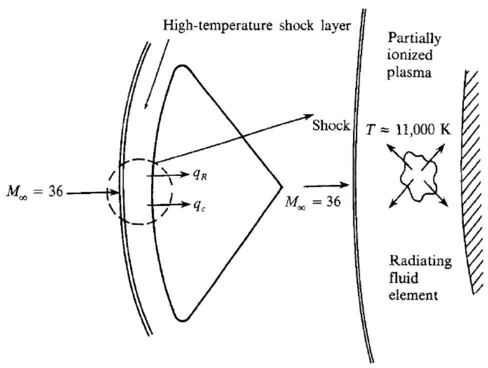


Fig. 4.7: Illustration of a high-temperature shock layer on a blunt body moving at hypersonic speeds

4.3 Hypersonic Shock wave Relations

The shock wave relations in the hypersonic flows are given in the figure. 4.8. Although the shock wave relations for hypersonic and supersonic flows are same, the rather different wave relations seen in figure 4.8 for hypersonic flows is basically through the assumption that at very hypersonic speeds, $M_1 \rightarrow \infty$ and $\gamma \rightarrow 1$.

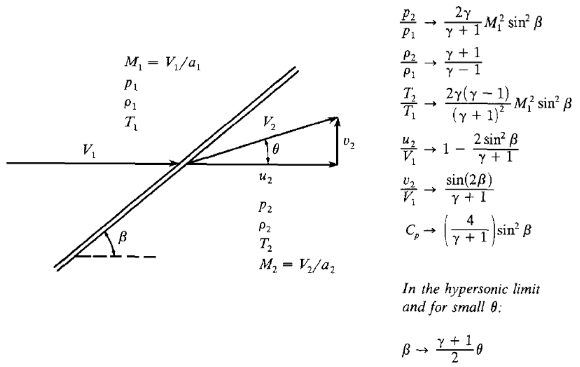


Fig. 4.8: Hypersonic Shock wave relations

For a completed detail about the derivation of the above relation refer to section 15.3 in "Modern compressible flow book".

4.4 Newtonian Theory

Here, we will obtain a simple expression for the pressure distribution over the surface of a blunt body. In Propositions 34 and 35 of his Principia, Isaac Newton considered that the force of impact between a uniform stream of particles and a surface is obtained from the loss of momentum of the particles normal to the surface. For example, consider a stream of particles with velocity V_∞, incident on a flat surface inclined at the angle ϑ with respect to the velocity, as shown in Fig. 4.9a. Upon impact with the surface, Newton assumed that the normal momentum of the particles is transferred to the surface, whereas the tangential momentum is preserved. Hence, after collision with the surface, the particles move along the surface, as sketched in Fig. 4.9a. The change in normal velocity is simply $V_\infty \sin\vartheta$. Now consider Fig. 4.9b. The mass flux of particles incident on a surface of area A is $\rho V_\infty A \sin\vartheta$.

Hence, the time rate of change of momentum of this mass flux, from Newton's reasoning, is

$$\textit{Mass flux} \times \textit{velocity change} \tag{4.17}$$

or

$$(\rho V_\infty A \sin\vartheta)(V_\infty \sin\vartheta) = \rho V_\infty^2 A \sin^2\vartheta \tag{4.18}$$

And in turn, from Newton's second law, this time rate of change of momentum is equal to the force F on the surface:

$$F = \rho V_\infty^2 A \sin^2\vartheta \tag{4.19}$$

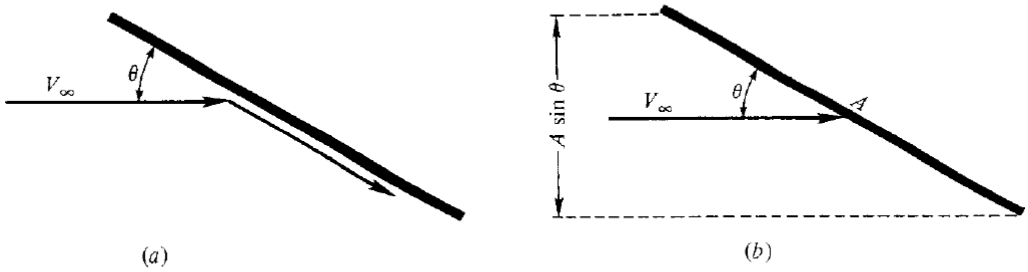


Fig. 4.9: Schematic for Newtonian Impact theory

In turn, the pressure is force per unit area, Which from the above equation is

$$\frac{F}{A} = \rho V_\infty^2 \sin^2\vartheta \tag{4.20}$$

Newton assumed the stream of particles in Fig. 4.9b to be linear, i.e., he assumed that the individual particles do not interact with each other, and have no random motion. Since modern science recognizes that static pressure is due to the random motion of the particles, and since Eq. 4.20 considers only the linear, directed motion of the particles, the value of F/A in Eq. 4.20 must be interpreted as the pressure difference above static pressure, namely, $F/A = p - p_\infty$. Therefore, from Eq. 4.20, and recalling from the definition of the pressure coefficient, $C_p = (p - p_\infty)/\frac{1}{2}\rho V_\infty^2$, we have

$$p - p_\infty = \rho V_\infty^2 \sin^2\vartheta \tag{4.21}$$

$$\frac{p - p_\infty}{\frac{1}{2}\rho V_2} = 2\sin^2\vartheta \tag{4.22}$$

$$\boxed{C_p = 2\sin^2\vartheta} \tag{4.23}$$

Equation 4.23 is te newtonian "sine-squared" law for preesure distribution on a surface inclined at an angle ϑ with respect to the freestream.

4.5 A Local surface Inclination Method: Modified Newtonian Theory

Lineari eds upersonic theory leads to a simple relation for the surface pressure coefficient, namely Equ. 4.14, repeated here:

$$C_p = \frac{2\vartheta}{\sqrt{M_\infty^2}} \tag{4.24}$$

Note from Eq. 4.14 that C_p, depends only on ϑ, the local surface inclination angle defined by the angle between a line tangent to the surface and the free-stream direction. In this sense, Eq. 4.14 is an example of a "local surface inclination method" for linearized supersonic flow. Question: Do any local surface inclination methods exist for hypersonic flow? The answer is yes. The oldest and most widely used of the hypersonic local surface inclination methods is newtonian theory.

recall the exact oblique shock relation for C_p from Figure. 4.8,

$$C_p = \frac{4}{\gamma + 1} \sin^2 \beta \tag{4.25}$$

Now consider the limit that as $M \rightarrow \infty$, $\gamma \rightarrow 1$, we can see the above equation becomes

$$C_p \rightarrow 2 \sin^2 \beta \tag{4.26}$$

Let us go further. Consider the exact oblique shock relation for ρ/ρ_∞, given by

$$\frac{\rho_2}{\rho_\infty} = \frac{(\gamma + 1) M_\infty^2 \sin^2 \beta}{(\gamma - 1) M_\infty^2 \sin^2 \beta + 2} \tag{4.27}$$

The above equation, as $M_\infty \rightarrow \infty$,

$$\frac{\rho_2}{\rho_\infty} \rightarrow \frac{\gamma + 1}{\gamma - 1} \tag{4.28}$$

In the additional limits as $\gamma \rightarrow 1$, we find

$$\frac{\rho_2}{\rho_\infty} \rightarrow \infty \tag{4.29}$$

Here the density behind the shock is infinitely large. In turn, mass flow considerations then dictate that the shock wave is coincident with the body surface. Now considering the $\vartheta - \beta - M$ relation,

$$\tan \vartheta = 2 \cot \beta \quad \frac{M_1^2 \sin^2 \beta - 1}{M_1^2 (\gamma + \cos 2\beta) + 2} \tag{4.30}$$

as β is small and M_1 is very large, the above reduces to

$$\frac{\beta}{\vartheta} = \frac{\gamma + 1}{2} \tag{4.31}$$

Now as $\gamma \rightarrow 1$ and $M_\infty \rightarrow \infty$,

$$\boxed{\vartheta = \beta} \tag{4.32}$$

i.e the shock wave lies on the body. In light of this result, the Equ. 4.26 can be written as

$$C_p = 2 \sin^2 \vartheta \tag{4.33}$$

In the newtonian model of fluid flow, the particles in the free stream impact only on the frontal area of the body; they cannot curl around the body and impact on the back surface. Hence, for that portion of a body which is in the ”shadow” of the incident flow, such as the shaded region sketched in Fig. 4.10, no impact pressure is felt. Hence, over this shadow region it is consistent to assume that $p = p_\infty$ and therefore $C_p = 0$, as indicated in Fig. 4.10.

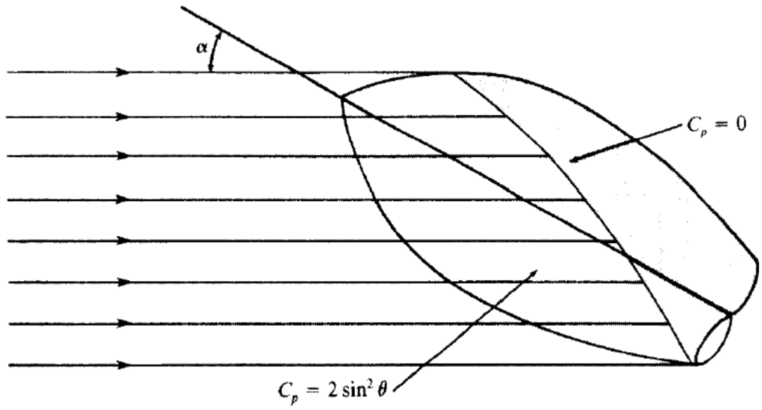


Fig. 4.10: Shadow region on the leeward side of a body, from newtonian theory.

4.6 Lift and Drag in Hypersonic flow

It is instructive to examine newtonian theory applied to a flat plate, as sketched in Fig. 4.11. Here, a two-dimensional flat plate with chord length c is at an angle of attack α to the free stream. Since we are not including friction, and because surface pressure always acts normal to the surface, the resultant aerodynamic force is perpendicular to the plate, i.e., in this case the normal force N is the resultant aerodynamic force. (For an infinitely thin flat plate, this is a general result which is not limited to newtonian theory, or even to hypersonic flow.) In turn, N is resolved into lift and drag, denoted by L and D, respectively, as shown in Fig. 4.11.

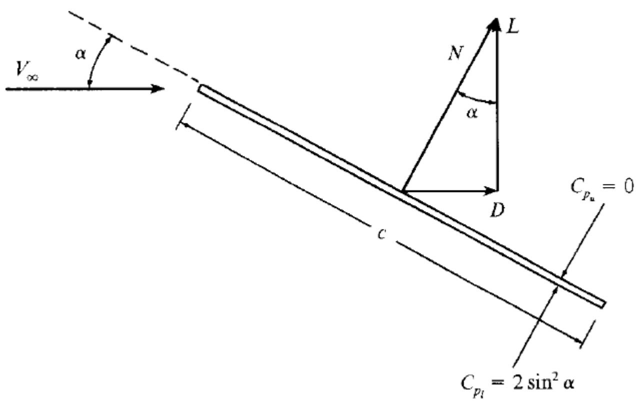


Fig. 4.11: Flat plate at angle of attack. Illustration of aerodynamic forces.

According to newtonian theory, the pressure coefficient on the lower surface is

$$C_{pl} = 2\sin^2\alpha \tag{4.34}$$

and that on the upper surface, which is in the shadow region, is

$$C_{p\alpha} = 0 \tag{4.35}$$

Defining the normal force coefficient as $c_n = N/q_\infty S$, where $S = (c)(l)$, we can readily calculate c_n, by integrating the pressure coefficients over the lower and upper surfaces

$$c_n = \frac{1}{c}\int_0^c (C_{pl} - C_{pu})\, dx \tag{4.36}$$

where x is the distance along the chord from the leading edge. Substituting Equs. 4.34 and 4.35 in the above equation, we get

$$c_n = \frac{1}{c}\, 2\sin^2\alpha\, c \tag{4.37}$$

$$c_n = 2\sin^2\alpha \tag{4.38}$$

From the geometry of Figure. 4.11, we can see that the lift and drag coefficients, defined as $c_l = L/q_\infty S$ and $c_d = D/q_\infty S$, respectively, where $S = (c)(l)$, are given by

$$c_l = c_n \cos\alpha \tag{4.39}$$

and

$$c_d = c_n \sin\alpha \tag{4.40}$$

Substituting, Equ. 4.38 into 4.39 and 4.40, we get

$$c_l = 2\sin^2\alpha\cos\alpha \tag{4.41}$$

and

$$c_d = 2\sin^3\alpha \tag{4.42}$$

Finally, the lift-to-drag ratio is given by

$$\frac{L}{D} = \cot\alpha \tag{4.43}$$

Chapter 5

Flow through nozzles and variable area ducts

5.1 Quasi one-dimensional flow

In the early chapters the one dimensional flow was strictly treated as constant area flow. In the present chapters, the restriction on constant area will be relaxed by allowing the streamtube area A to vary with distance x as shown in the fig 5.1. At the same time we will continue to assume that all flow properties are uniform across any given cross section of the flow, and hence are functions of *x* only (and time *t* if the flow is unsteady). Such a flow where A = A(x), p = p(x), ρ = ρ(x) and V = u = u(x) for steady flow is defined as *Quasi-one-dimensional flow*.

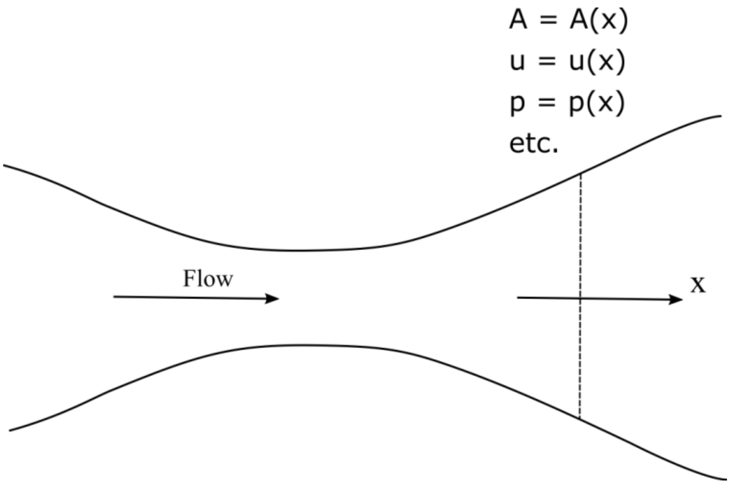


Fig. 5.1: Quasi one-dimensional flow

5.2 Governing Equations

Algebraic equations for steady quasi-one-dimensional flow can be obtained by applying the integral form of the conservation equations to the variable-area control volume sketched in Figure 5.2.

5.2.1 Continuity Equation

The continuity equation is repeated here,

$$-\oiint_S \rho V.dS = \frac{\partial}{\partial t}\iiint_v \rho dV \tag{5.1}$$

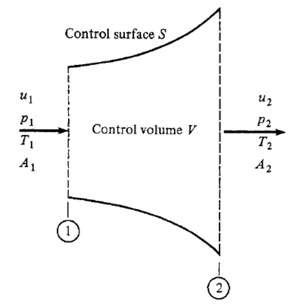


Fig. 5.2: Finite control volume for quasi one-dimensional flow

when integrated over the control volume in Figure 5.2, for steady flow, directly to

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \tag{5.2}$$

5.2.2 Momentum Equation

The integral form of the momentum equation is

$$\oiint_S (\rho V.dS)\, V + \iiint_V \frac{\partial (\rho V)}{\partial t} dV = \iiint_V \rho f dV - \oiint_S p dS \tag{5.3}$$

Applied to Figure 5.2, assuming steady flow and no body forces, it directly becomes

$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p dA = p_2 A_2 + \rho_2 u_2^2 A_2 \tag{5.4}$$

This is the momentum equation for steady quasi-one-dimensional flow. Note that it is not strictly an algebraic equation because of the integral term which represents the pressure force on the sides of the control surface between locations 1 and 2.

5.2.3 Energy Equation

The integral form of the energy equation is

$$\iiint_V \dot{q}\rho dV - \oiint_S pV.dS + \iiint_V \rho (f.V)\, dV = \iiint_V \frac{\partial}{\partial t} \rho \left(e + \frac{V^2}{2}\right) dV + \oiint_S \rho \left(e + \frac{V^2}{2}\right) V.dS \tag{5.5}$$

Applied to Figure 5.2 and assuming steady adiabatic flow with no body forces, it directly yields

$$-(-p_1 u_1 A_1 + p_2 u_2 A_2) = \rho_1 \left(e_1 + \frac{u_1^2}{2}\right) = p_2 u_2 A_2 + \rho_2 u_2 A_2 \left(e_2 + \frac{u_2^2}{2}\right) u_2 A_2 \tag{5.6}$$

Rearranging,

$$p_1 u_1 A_1 + \rho_1 u_1 A_1 \left(e_1 + \frac{u_1^2}{2}\right) = p_2 u_2 A_2 + \rho_2 u_2 A_2 \left(e_2 + \frac{u_2^2}{2}\right) \tag{5.7}$$

Dividing the above equation with 5.2, we get

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2} \tag{5.8}$$

Noting that $h = e + p/\rho$, The above equation becomes

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{5.9}$$

This is the energy equation for steady adiabatic quasi-one-dimensional flow-it states that the total enthalpy is constant along the flow.

5.3 Euler's Equation for Quasi 1D flow

Consider an infinitesimal control volume with a variable are cross section as shown in Figure 5.3.

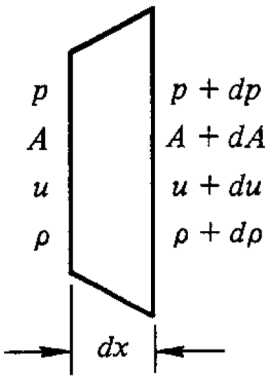


Fig. 5.3: Infinitesimal control volume

The Quasi one dimensional continuity equation Equ. 5.2 can be written as

$$\rho u A = constant \tag{5.10}$$

Hence,

$$d(\rho u A) = 0 \tag{5.11}$$

To obtain a differential form of the momentum equation, apply Equ. 5.4 to the infinitesimal control volume sketched in Figure 5.3, where the length in the x direction is dx:

$$pA + \rho u^2 A + pdA = (p + dp)(A + dA) + (\rho + d\rho)(u + du)^2 (A + dA) \tag{5.12}$$

Dropping all second-order terms involving products of differentials, this becomes

$$Adp + Au^2 d\rho + \rho u^2 dA + 2\rho u A du = 0 \tag{5.13}$$

Expanding Equ. 5.11, and multiplying by u,

$$\rho u^2 dA + \rho u A du + Au^2 d\rho = 0 \tag{5.14}$$

Substracting the above equation from Equ. 5.13, we get

$$\boxed{dp = -\rho u du} \tag{5.15}$$

Equ. 5.15 is called the Euler's equation. Finally, a differntial form of the energy equation is obtained from Equ. 5.9,which states that

$$h + \frac{u^2}{2} = const \tag{5.16}$$

Hence,

$$\boxed{dh + udu = 0} \tag{5.17}$$

5.4 Area Velocity Relation

From the differential form of 1D continuity equation, Equ. 5.11 can be written as

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{5.18}$$

To eliminate $d\rho/\rho$ from Equ. 5.18, consider the Euler's equation Equ. 5.15 i.e:

$$\frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -udu \tag{5.19}$$

Recall that we are considering adiabatic, inviscid flow, i.e., there are no dissipative mechanisms such as friction, thermal conduction, or diffusion acting on the flow. Thus, the flow is isentropic. Hence, any change in pressure, dp, in the flow is accompanied by a corresponding isentropic change in density, $d\rho$. Therefore, we can write

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2 \tag{5.20}$$

Combing Equ. 5.19 and Equ. 5.20, we get

$$a^2\frac{d\rho}{\rho} = -udu \tag{5.21}$$

or

$$\frac{d\rho}{\rho} = -\frac{udu}{a^2} = -\frac{u^2du}{a^2u} = -M^2\frac{du}{u} \tag{5.22}$$

Substituting the above equation in Equ. 5.18, we get

$$\boxed{\frac{dA}{A} = \left(M^2 - 1\right)\frac{du}{u}} \tag{5.23}$$

Equation. 5.23 is an important relation. It is called the area-velocity relation, and it tells us this information:

1. For $M \to 0$, which in the limit corresponds to incompressible flow, Equ. 5.23 shows that $Au = const$. This is the familiar continuity equation for the incompressible flow.

2. For $0 \leq M < 1$ (subsonic flow), an *increase* in velocity (positive du)is

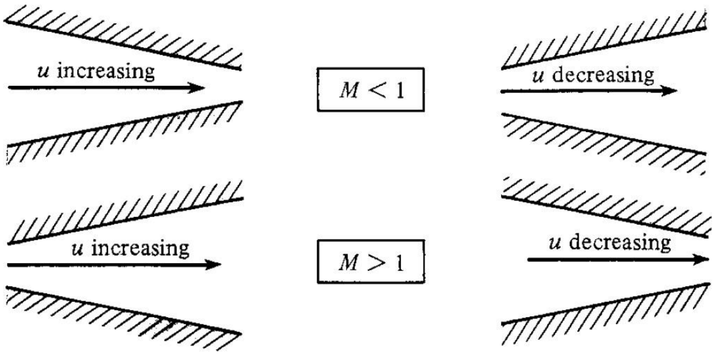


Fig. 5.4: Flow in coverging and diverging ducts

associated with a *decrease* in area (negative dA), and vice versa. Therefore, the familiar result from incompressible flow that the velocity increases in a converging duct and decreases in a diverging duct still holds true for subsonic compressible flow (see top of Figure. 5.4)

3. For $M > 1$ (supersonic flow), an increase in velocity is associated with an increase in area, and vice versa. Hence, we have a striking difference in comparison to subsonic flow. For supersonic flow, the velocity increases in a diverging duct and decreases in a converging duct (see bottom of Figure. 5.4).

4. For $M = 1$ (sonic flow), Equ. 5.23 yields $dA/A = 0$, which mathematically corresponds to a minimum or maximum in the area distribution. The minimum in area is the only physically realistic solution, as described next.

These results clearly show that for a gas to expand isentropically from subsonic to supersonic speeds, it must flow through a convergent-divergent duct (or stream-tube), as sketched at the top of Figure. 5.4. Moreover, at the minimum area that divides the convergent and divergent sections of the duct, we know from item 4 above that the flow must be sonic. This minimum area is called a throat. Conversely, for a gas to compress isentropically from supersonic to subsonic speeds, it must also flow through a convergent-divergent duct, with a throat where sonic flow occurs, as sketched at the bottom of Figure. 5.4. From this discussion, we recognize why rocket engines have large, bell-like nozzle shapes as sketched in Figure. 5.6-to expand the exhaust gases to high-velocity, supersonic speeds.

5.5 Area-Mach Number relation for flow inside Nozzles

Consider the duct shown in Figure. 5.7.

At the throat, the flow is sonic. Hence, denoting conditions at sonic speed by an asterisk, we have, at the throat, $M^* = 1$ and $u^* = a^*$. The area of the throat is A^*. At any other section of the duct, the local area, Mach number, and velocity are A, M and u, respectively. Apply Equ. 5.2 between these two locations:

$$\rho^* u^* A^* = \rho u A \tag{5.24}$$

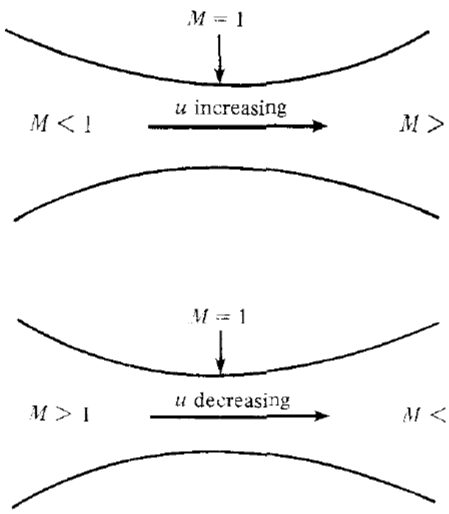


Fig. 5.5: Flow in coverging and diverging ducts

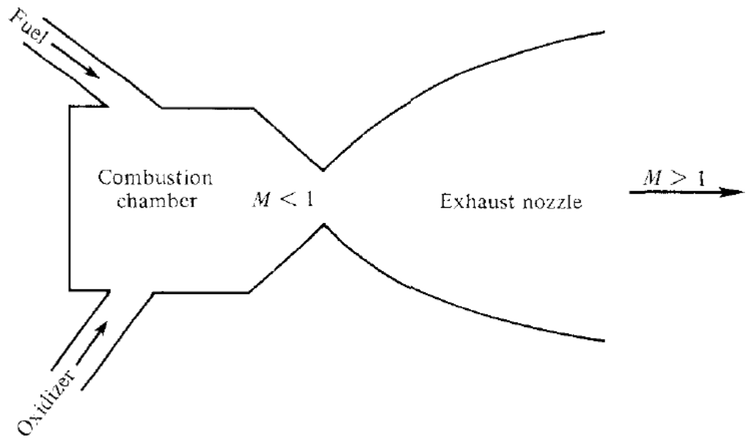


Fig. 5.6: Flow in coverging and diverging ducts

Since $u^* = a^*$, The above equation becomes

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u} \tag{5.25}$$

where ρ_o is the stagnation density and is defined throughout the isentropic flow. Recalling the isentropic relation between stagnation density and staic density at any point in the flow,

$$\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{1/(\gamma-1)} \tag{5.26}$$

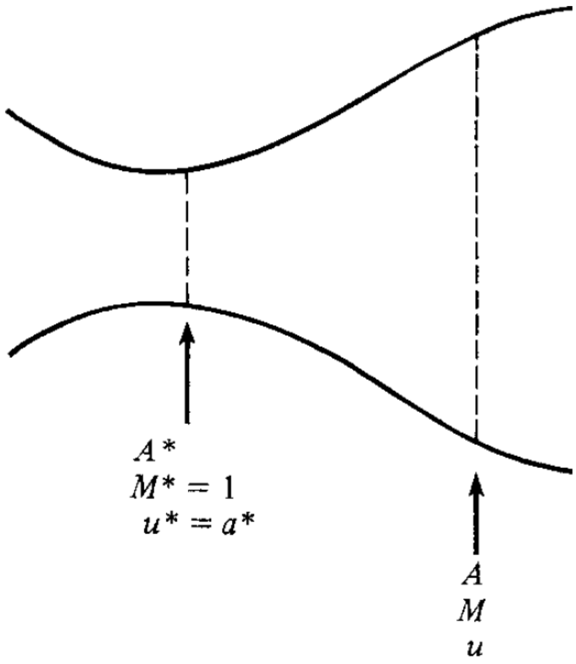


Fig. 5.7: Flow in coverging and diverging ducts

And apply the above equation to sonic condition where $M = 1$, we have

$$\frac{\rho_o}{\rho^*} = \left(\frac{\gamma + 1}{2}\right)^{1/(\gamma-1)} \tag{5.27}$$

Also recalling the relation between the characteristic Mach number M^* and the freestream Mach number M,

$$\left(\frac{u}{a^*}\right)^2 = M^{*2} = \frac{\frac{\gamma+1}{2}M^2}{1 + \frac{\gamma-1}{2}M^2} \tag{5.28}$$

Squaring Equ. 5.25 and substituting Equs. 5.27, 5.28 and 5.26, we get

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{\rho^*}{\rho_o}\right)^2 \left(\frac{\rho_o}{\rho}\right)^2 \left(\frac{a^*}{u}\right)^2 \tag{5.29}$$

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)} \left(1 + \frac{\gamma-1}{2}M^2\right)^{2/(\gamma-1)} \left(\frac{1 + \frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}M^2}\right) \tag{5.30}$$

$$\boxed{\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}} \tag{5.31}$$

Equation 5.31 is called the *area-Mach number relation*, and it contains a striking result. Turned inside out, Equ. 5.31 tells us that $M = f\,(A/A^*)$, i.e., the Mach number at any location in the duct is a function of the ratio of the local duct area to the sonic throat area. As seen from Equ. 5.23, A must be greater than or at least equal to A^*; the case where $A < A^*$ is physically not possible in an isentropic flow. Also, from Equ. 5.31 there are two values of M that correspond to a given $A/A^* > 1$, a subsonic and a supersonic value. The solution of Equ. 5.31 is plotted in Figure 5.8, which clearly delineates the subsonic and supersonic branches.

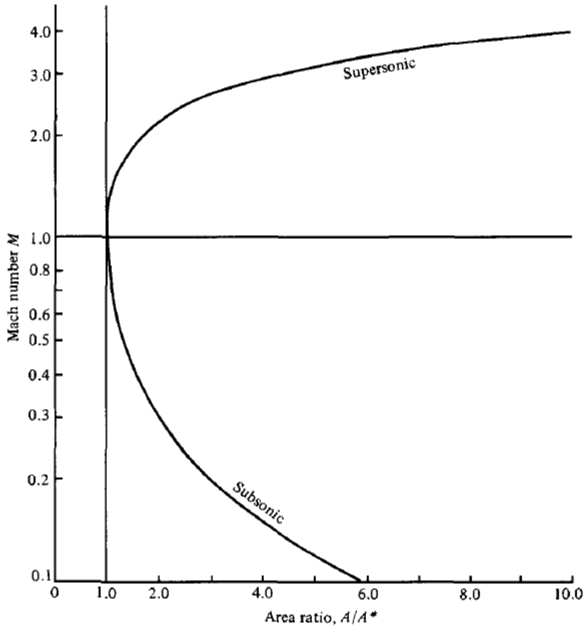


Fig. 5.8: Flow in coverging and diverging ducts

5.6 Isentropic Flow through Convergent Divergent Nozzles - Choked Flow, Under-expansion and Over-expansion

Consider a given convergent-divergent nozzle, as sketched in Figure. 5.9. Assume that the area ratio ratio at the inlet A_i/A^* is very large, $A_i/A^* \rightarrow \infty$, and that the inlet is fed with gas from a large reservoir at pressure and temperature p_o and T_o, respectively.Because of the large inlet area ratio, $M \approx 0$, hence p_o and T_o are essentially stagnation (or total values). (The Mach number cannot be precisely zero in the reservoir or else there would be no mass flow through the nozzle. It is a finite value, but small enough to assume that it is essentially zero.). Furthermore, assume that the given convergent-divergent nozzle expands the flow isentropically to supersonic speeds at the exit. For the given nozzle, there is only one possible isentropic solution for supersonic flow, and Equ. 5.31 is the key to this solution. In the convergent portion of the nozzle, the subsonic flow is accelerated, with the subsonic value of M dictated by the local value of A/A^* as given by the lower branch of Figure. 5.8. The consequent variation of Mach number with distance x along the nozzle is sketched in Figure. 5.9. At the throat, where the throat area $A = A^*$, $M = 1$. In the divergent portion of the nozzle, the flow expands supersonically, with the supersonic value of M dictated by the local value of A/A^* as given by the upper branch of Figure. 5.8. This variation of M with x in the divergent nozzle is also sketched in Figure. 5.9. Once the variation of Mach number through the nozzle is known, the variations of static temperature, pressure, and density follow from Isentropic relations. The resulting variations of p and T are shown in Figure. 5.9, respectively. Note that the pressure, density, and temperature decrease continuously throughout the nozzle. Also note that the exit pressure, density, and temperature ratios, p_e/p_o, ρ_e/ρ_o, and T_e/T_o, depend only on the exit area ratio, A_e/A^* via Equ. 5.31.

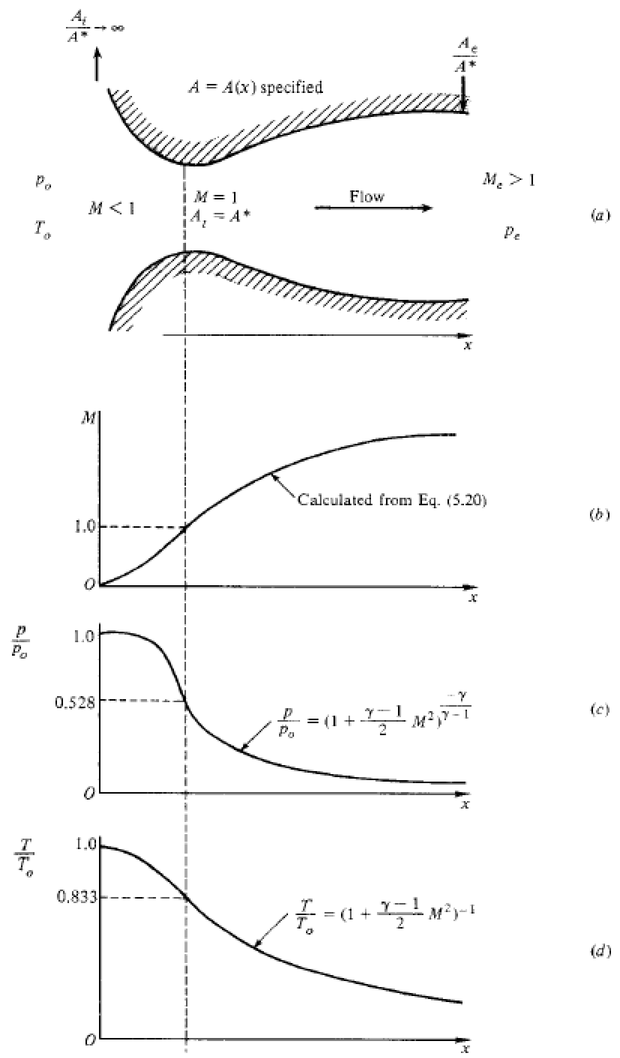


Fig. 5.9: Flow in coverging and diverging ducts

5.7 The Effect of Different Pressure Ratios Across a Given Nozzle

If a convergent-divergent nozzle is simply placed on a table, and nothing else is done, obviously nothing is going to happen; the air is not going to start rushing through the nozzle of its own accord. To accelerate a gas, a pressure difference must be exerted, as clearly stated by Euler's equation, Equ. 5.15. Therefore, in order to establish a flow through any duct, the exit pressure must be lower than the inlet pressure, i.e., $p_e/p_o < 1$. Indeed, for completely shockfree isentropic supersonic flow to exist in the nozzle of Figure. 5.9a, the exit pressure ratio must be precisely the value of p_e/p_o shown in Figure. 5.9c.

What happens when p,/p, is not the precise value as dictated by Fig. 5.9? In other words, what happens when the backpressure downstream of the nozzle exit is independently governed (say by exhausting into an infinite reservoir with controllable pressure)? Consider a convergent-divergent nozzle as sketched in Fig. 5.10a. Assume that no flow exists in the nozzle, hence $p_e = p_o$. Now assume that p_e is minutely reduced below p_o. This small pressure difference will cause a small wind to blow through the duct at low subsonic speeds. The local Mach number will increase slightly through the convergent portion of the nozzle, reaching a maximum at the throat, as shown by curve 1 of Fig. 5.10b. This maximum will not be sonic; indeed it will be a low subsonic value. Keep in mind that the value A^* defined earlier is the sonic throat area, i.e., that area where $M = 1$. In the case we are now considering, where $M < 1$ at the minimum-area section of the duct, the real throat area of the duct, A_t, is larger than A^*, which for completely subsonic flow takes on the character of a reference quantity different from the actual geometric throat area. Downstream of the throat, the subsonic flow encounters a diverging duct. and hence M decreases as shown in Fig. 5.10b. The corresponding variation of static pressure is given by curve 1 in Fig. 5.10c. Now assume p_e is further reduced. This stronger pressure ratio between the inlet and exit will now accelerate the flow more, and the variations of subsonic Mach number and static pressure through the duct will be larger. as indicated by curve 2 in Figs. 5.10b and c. If p_e is further reduced, there will be some value of p_e at which the flow will just barely go sonic at the throat, as given by the curve 3 in Figs. 5.10b and c. In this case, $A_t = A^*$. Note that all the cases sketched in Figs 5.10b and c are subsonic flows. Hence, for subsonic flow through the convergent-divergent nozzle shown in Fig. 5.10a, there are an infinite number of isentropic solutions, where both p_e/p_o and A/A_t are the controlling factors for the local flow properties at any given section. This is a direct contrast with the supersonic case discussed earlier, where only one isentropic solution exists for a given duct, and where AIA* becomes the only controlling factor for the local flow properties (relative to reservoir properties).

For the cases shown in Figs. 5.10a, b, and c, the mass flow through the duct increases as p, decreases. This mass flow can be calculated by evaluating Eq. (5.1) at the throat, $m = \rho_t A_t u_t$. When p_e is reduced to p_{e3}, where sonic flow is attained at the throat, then $m = p^* A^* a^*$. If p, is now reduced further, $p, < p_{e3}$, the Mach number at the throat cannot increase beyond $M = 1$; this is dictated by Equ. 5.31. Hence, the flow properties at the throat, and indeed throughout the entire subsonic section of the duct, become "frozen" when $p_e < p_{e3}$, i.e., the subsonic flow becomes unaffected and the mass flow remains constant for $p_e < p_{e3}$. This condition, after sonic flow is attained at the throat, is called choked flow. No matter how low p_e is made, after the flow becomes choked, the mass flow remains constant. This phenomenon is illustrated in Fig. 5.11. Note that sonic flow at the throat corresponds to a pressure ratio $p^*/p_o = 0.528$ for $\gamma = 1.4$; however, because of the divergent duct downstream of the throat, the value of p_{e3}/p_o required to attain sonic flow at the throat is larger than 0.528, as shown in Figs. 5.10c and 5.11.

What happens in the duct when p, is reduced below p_{e3}? In the convergent portion, as we stated, nothing happens. The flow properties remain as given by the subsonic portion of curve 3 in Fig. 5.10b and c. However, a lot happens in the

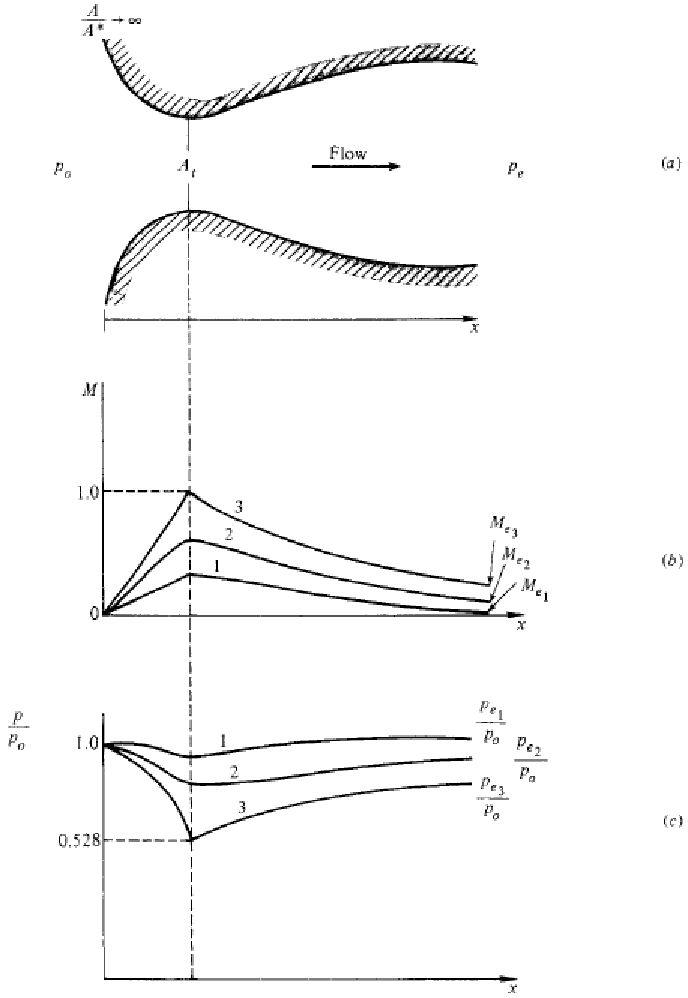


Fig. 5.10: Subsonic Flow in a coverging-diverging Nozzle

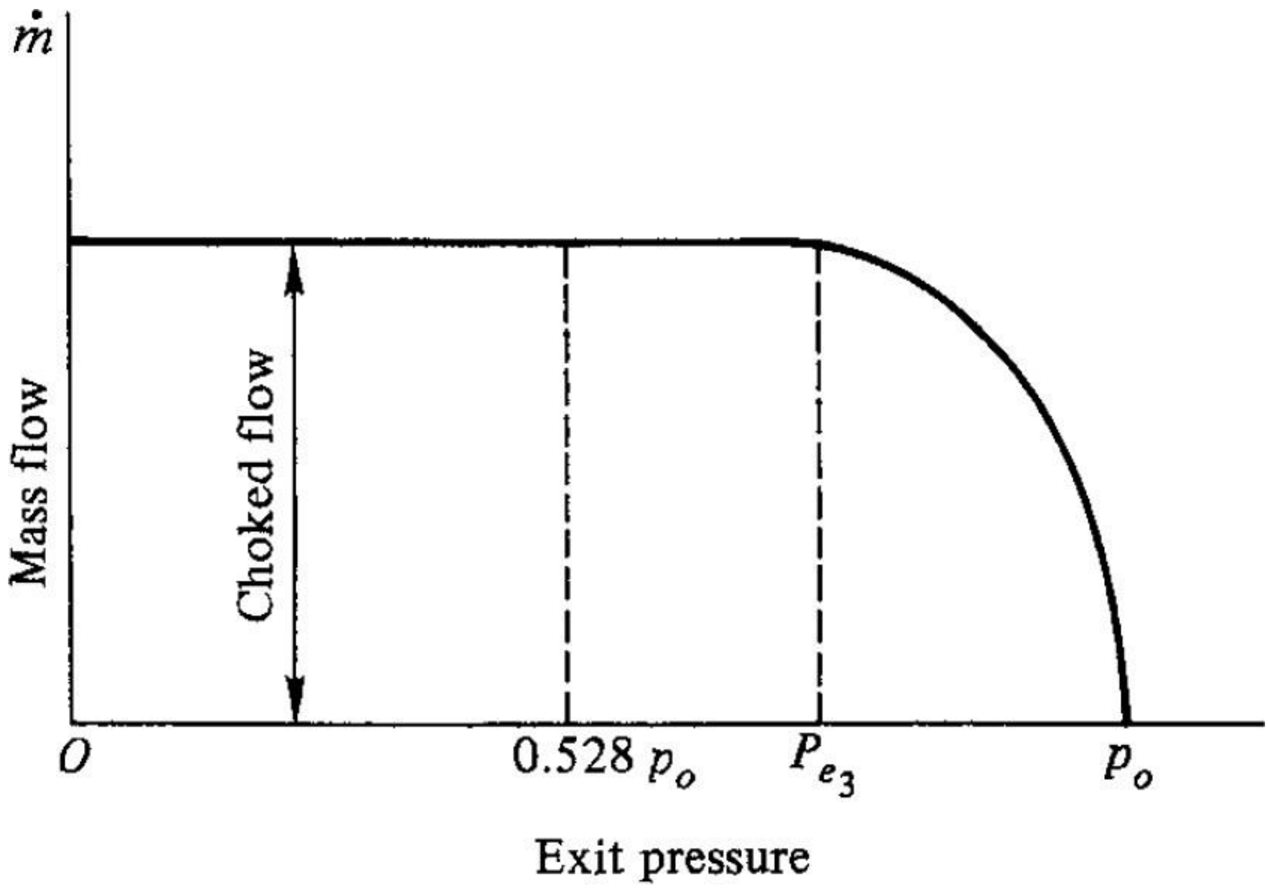


Fig. 5.11: Variation of mass flow with exit pressure; illustration of choked flow.

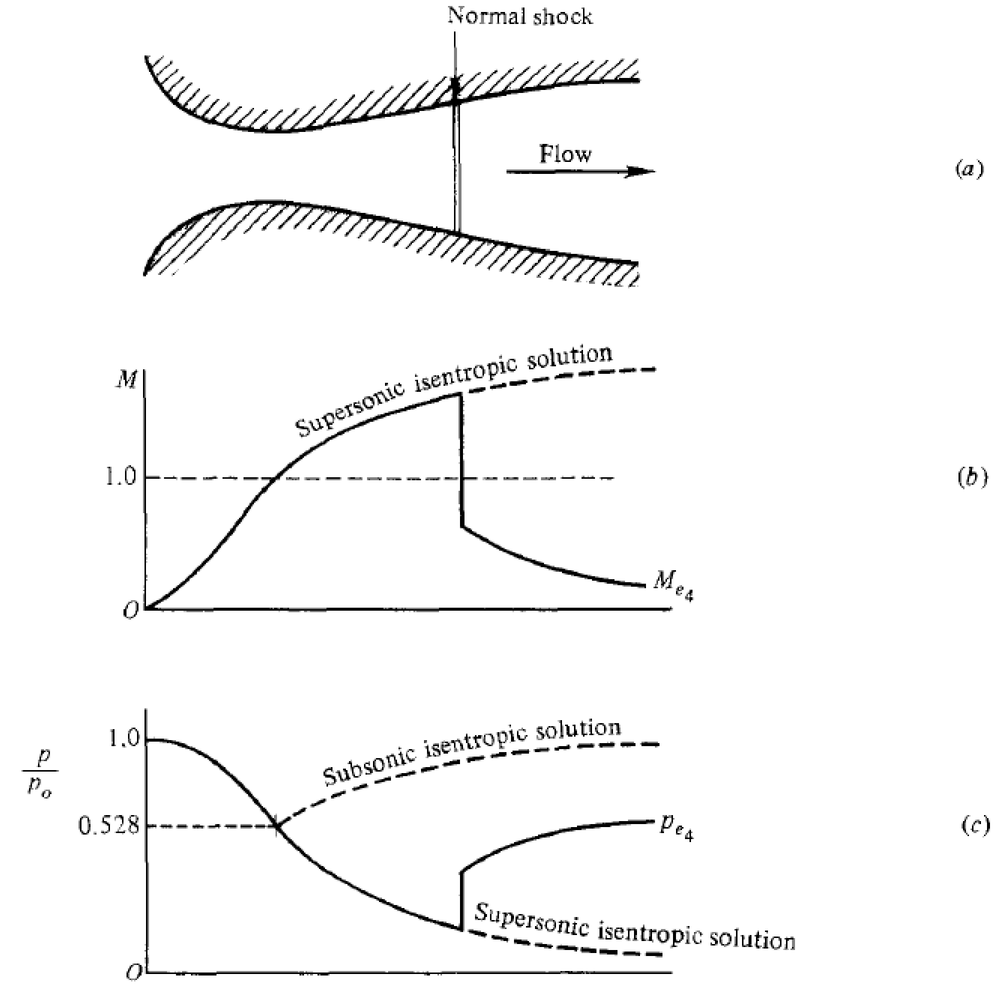


Fig. 5.12: Flow with a shock wave inside a convergent-divergent nozzle

divergent portion of the duct. No isentropic solution is allowed in the divergent duct until p_e is adequately reduced to the specified low value dictated by Fig. 5.9c. or values of exit pressure above this, but below p_{e3}, a normal shock wave exists inside the divergent duct. This situation is sketched in Fig. 5.12. Let the exit pressure be given by p_{e4} .

There is a region of supersonic flow ahead of the shock. Behind the shock, the flow is subsonic, hence the Mach number decreases towards the exit and the static pressure increases to p_e at the exit. The location of the normal shock wave in the duct is determined by the requirement that the increase of static pressure across the wave plus that in the divergent portion of the subsonic flow behind the shock be just right to achieve p_{e4} at the exit. As the exit pressure is reduced further, the normal shock wave will move downstream, closer to the nozzle exit. It will stand precisely at the exit when $pe = p_{e5}$, where p_{e5} is the static pressure behind a normal shock at the design Mach number of the nozzle. This is illustrated in Figs. 5.13a, b, and c. In Fig. 5.13c, p_{e6} represents the proper isentropic value for the design exit Mach number, which exists immediately upstream of the normal shock wave standing at the exit. When the downstream backpressure p_B is further decreased such that $p_{e6} < p_B < p_{e5}$, the flow inside the nozzle is fully supersonic and isentropic, with the behavior the same as given earlier in Figs. 5.9a, b, c, and d. The increase to the backpressure takes place across an oblique shock attached to the nozzle exit, but outside the duct itself. This is sketched in Fig. 5.13d. If the backpressure is further reduced below p_{e6}, equilibration of the flow takes place across expansion waves outside the duct, as shown in Fig. 5.13e.

When the situation in Fig. 5.13d exists, the nozzle is said to be overexpanded, because the pressure at the exit has expanded below the back pressure, $p_{e6} < p_B$. Conversely, when the situation in Fig. 5.13e exists, the nozzle is said to be underexpanded, because the exit pressure is higher than the back pressure, $p_{e6} > p_B$ and hence the flow is capable of additional expansion after leaving the nozzle.

5.8 Diffusers

Difusers are the devices used to slow the flow with as samall a loss of total pressure as possible Let us go through a small thought experiment. Assume that we want to design a supersonic wind tunnel with a test section Mach number of 3. Some immediate information about the nozzle is obtained from isentropic property tables; at $M = 3$, $A_e/A^* = 4.23$ and $p_o/p_e = 36.7$. Assume the wind tunnel exhausts to the atmosphere. What value of total pressure p_o must be provided by the reservoir to drive the tunnel? There are several possible alternatives. The first is to simply exhaust the nozzle directly to the atmosphere, as sketched in Fig. 5.14.

In order to avoid shock or expansion waves in the test region downstream of the exit, the exit pressure p_e, must be equal to the surrounding atmospheric pressure, i.e., $p_e = 1$ atm. Since $p_o/p_e = 36.7$, the driving reservoir pressure for this case must be 36.7 atm. However, a second alternative is to exhaust the nozzle into a constant-area duct which serves as the test section, and to exhaust this duct into the atmosphere, as sketched in Fig. 5.15. In this case, because the testing area is inside the duct, shock waves from the duct exit will not affect the test section. Therefore, assume a normal shock stands at the duct exit. The static pressure behind the normal shock is p_2, and because the flow is subsonic behind the shock, $p_2 = p_\infty = 1$ atm. In this case, the reservoir pressure p_o is obtained from

$$p_o = \frac{p_o}{p_e}\frac{p_e}{p_2} p_\infty = 367 \frac{1}{10.33} 1 = 3.55 atm \tag{5.32}$$

where p_2/p_e is the static pressure ratio across a normal shock at Mach 3, obtained from Normal shock table. Note that, by the simple addition of a constant-area duct with a normal shock at the end, the reservoir pressure required to drive the wind tunnel has markedly dropped from 36.7 to 3.55 atm. Now, as a third alternative, add a divergent duct behind the normal shock in Fig. 5.15 in order to slow the already subsonic flow to a lower velocity before exhausting to the atmosphere. This is sketched in Fig. 5.16. At the duct exit, the Mach number is a very low subsonic value, and for all practical purposes the local total and static pressure are the same. Moreover, assuming an isentropic flow in the divergent duct behind the shock, the total pressure at the duct exit is equal to the total pressure behind the normal shock. Consequently, $p_{o2} \approx p_\infty = 1$atm. From the

(a)

(b)

(c)

(d)

(e)

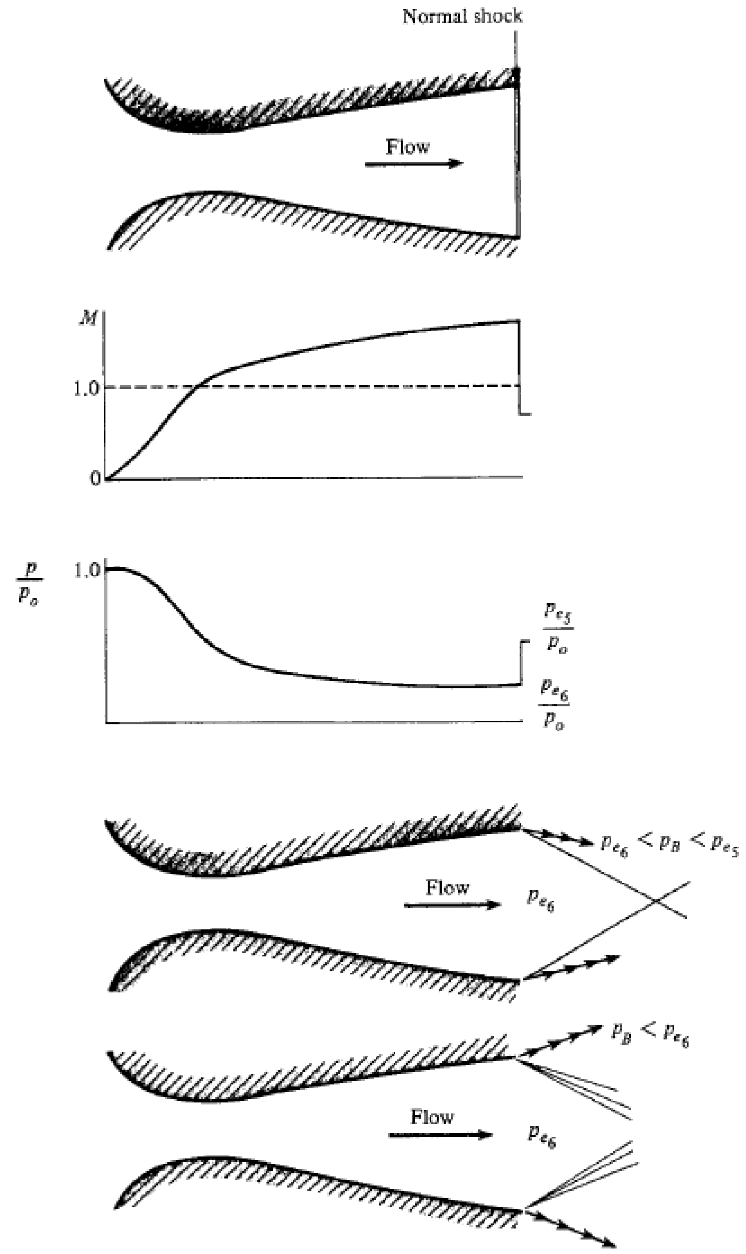


Fig. 5.13: Flow with shock and expansion waves at the exit of a convergent-divergent nozzle

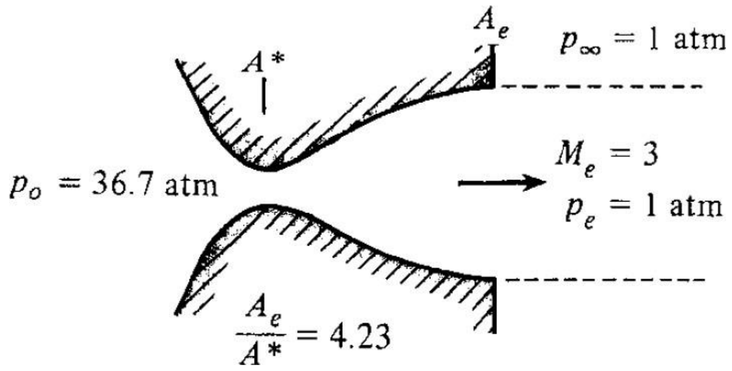


Fig. 5.14: Nozzle exhausting directly to atmosphere

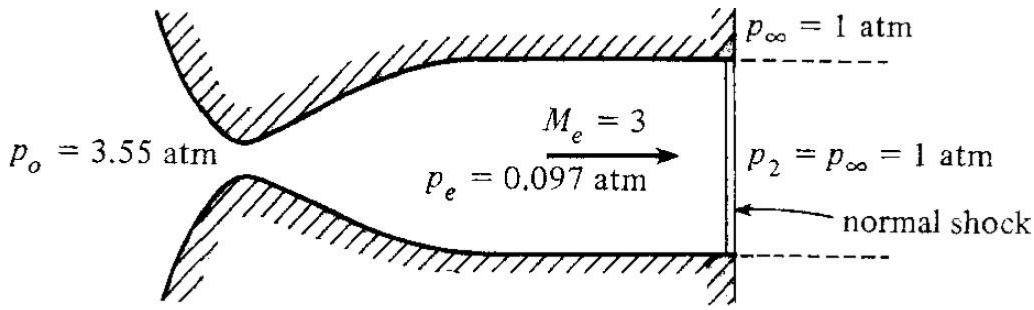


Fig. 5.15: Nozzle with a normal shock at the exit, exhausting to the atmosphere

normal shock tables, the Mach number behind the shock is $M_2 = 0.475$, and the ratio of total to static pressure at this Mach number (from isentropic property tables) is $p_{o2}/p_2 = 1.17$, Hence

$$p_o = \frac{p_o}{p_e}\frac{p_e}{p_2}\frac{p_2}{p_{o2}} p_\infty = 36.7 \frac{1}{10.33}\frac{1}{1.17} 1 = 3.04 atm \tag{5.33}$$

5.9 Wave Reflection from a free boundary

The gas jet from a nozzle which exhausts into the atmosphere has a boundary surface which interfaces with the surrounding quiescent gas. The oblique shock waves shown in Fig. 5.13d and the expansion waves sketched in Fig. 5.13e must reflect from the jet boundary in such a fashion as to preserve the pressure at the boundary downstream of the nozzle exit. This jet boundary is not a solid surface as treated earlier; rather, it is a free boundary which can change in size and direction. Considering the overexpanded nozzle flow in Fig. 5.13d, the flow pattern downstream of the nozzle exit will appear as sketched in Fig. 5.17. The various reflected waves form a diamond-like pattern throughout the exhaust jet. Such a diamond wave pattern is visible in the exhaust from the free jet.

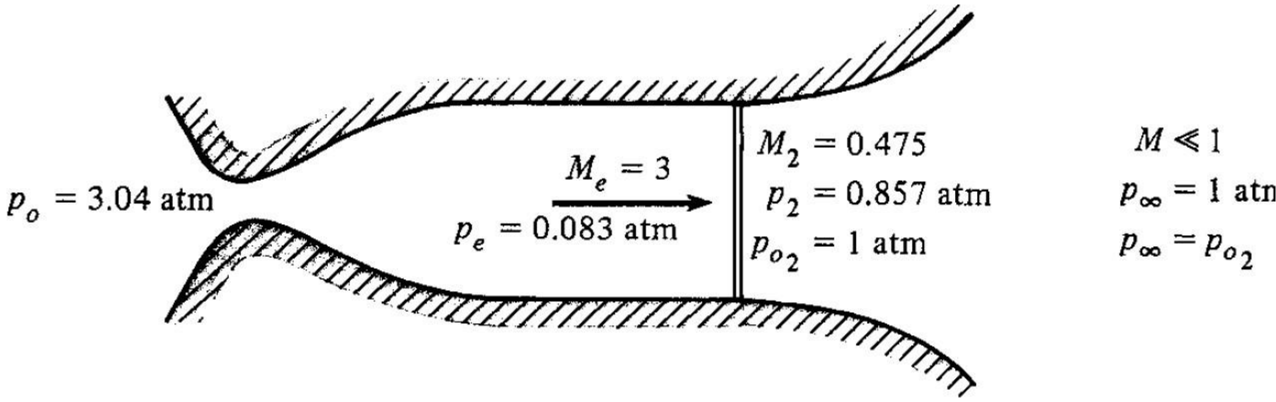


Fig. 5.16: Nozzle with a normal-shock diffuser. The normal shock is slightly upstream of the divergent duct.

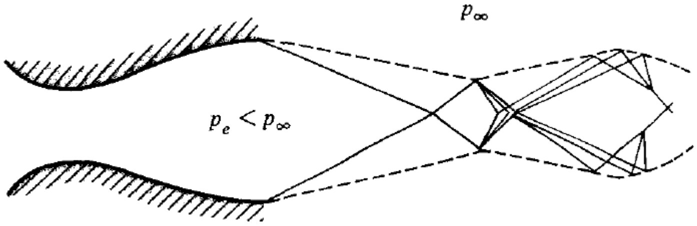


Fig. 5.17: Schematic of the diamond wave pattern in the exhaust from a supersonic nozzle

5.10 Supersonic Wind Tunnel

5.10.1 Parts of supersonic wind tunnel

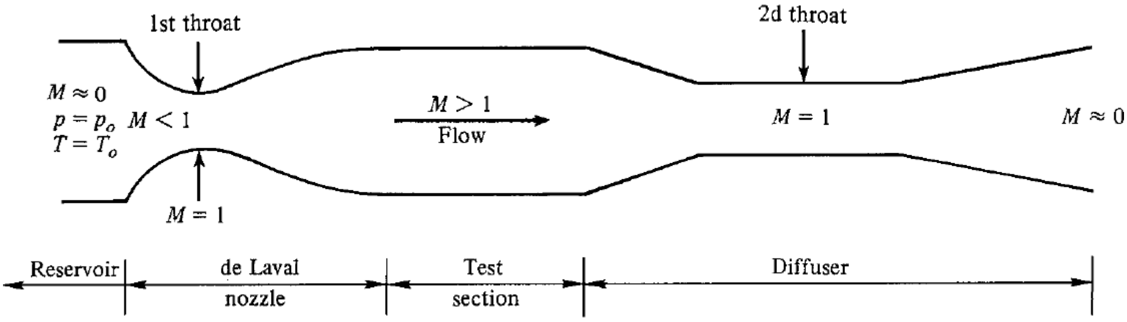


Fig. 5.18: Schematic of a supersonic Wind tunnel

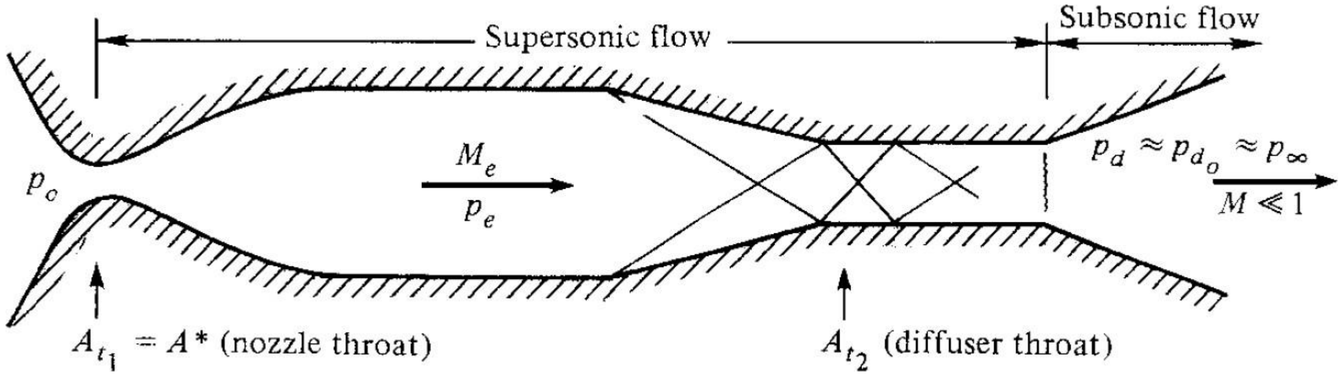


Fig. 5.19: Norh with a conventional supersonic diffuser

5.11 Numerical Problems

1. Consider the isentropic subsonic-supersonic flow through a convergent-divergent nozzle. The reservoir pressure and temperature are 10 atm and 300 K, respectively. There are two locations in the nozzle where $A/A^* = 6$: one in the convergent section and the other in the divergent section. At each location, calculate M. p, T, and u.

   Solution: In the convergent section, the flow is subsonic. From the front of Table A.1, for subsonic flow with $A/A^* = 6$:

$$\boxed{M = 0.097}, \quad \frac{p_o}{p} = 1.006, \quad \frac{T_o}{T} = 1.002$$

Hence,

$$p = \frac{p}{p_o} p_o = \frac{1}{1.006} \times 10 = 9.94\ atm$$

$$T = \frac{T}{T_o} T_o = \frac{1}{1.002} \times 300 = \boxed{299.4\ K}$$

$$a = \sqrt{\gamma R T} = \sqrt{1.4 \times 287 \times 299.4} = 346.8\ m/s$$

$$u = Ma = 0.097 \times 346.8 = \boxed{33.6\ m/s}$$

In the divergent section, the flow is supersonic. From the supersonic section of Table A.1, for $A/A^* = 6$:

$$\frac{p_o}{p} = 63.13, \text{ and } \frac{T_o}{T} = 3.269$$

$$p = \frac{p}{p_o}p_o = \frac{1}{63.13} \times 10 = \boxed{0.1584\ atm}$$

$$T = \frac{T}{T_o}T_o = \frac{1}{3.269} \times 300 = \boxed{91.77\ K}$$

$$a = \sqrt{\gamma RT} = \sqrt{1.4 \times 287 \times 91.77} = 192.0\ m/s$$

$$u = Ma = 3.368 \times 192.0 = \boxed{646.7\ m/s}$$

2. A supersonic wind tunnel is designed is designed to produce flow at Mach 2.4. at standard atmospheric conditions. Calculate (i) the exit to throat area ratio of the nozzle (ii) Reservoir pressure and temperature.

   Solution: From Table A.1, for $M_e = 2.5$:

$$\frac{A_e}{A^*} = 2.637, \frac{p_o}{p_e} = 17.09, \frac{T_o}{T_e} = 2.25$$

   At standard sea level conditions, $p_e = 1\ atm$ and $T_e = 288\ K$, Hence,

$$p_o = \frac{p_o}{p_e}p_e = 17.09 \times 1 = \boxed{17.09\ atm}$$

$$T_o = \frac{T_o}{T_e}T_e = 2.25 \times 288 = \boxed{648\ K}$$